

UNIVERSITY OF HELSINKI

MASTER'S THESIS

Numerical computation of the module of a quadrilateral

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March 12, 2007

PREFACE

Writing this thesis has been a long and hard struggle. Considering all the difficulties I had with this project I have tried to write a simple introduction to the module of a quadrilateral, even if nobody ever reads this. I could have put more effort to improve the readability of the text but unfortunately all projects must come to an end and this project was no exception. This project ended because it needed to end not because it reached its natural conclusion.

This project would have never been completed without God's help in the darkest hours of this work. Therefore I want to give Him all the credit, and my eternal gratitude for completing this work

Furthermore, I thank my supervisors prof. Matti Vuorinen and Ph. D. Antti Rasila for introducing me to this subject and for their suggestions and corrections to the work itself. Also I thank Mr. Kai Mäkelä for helping me with the English spelling.

Finally, I would like to thank my parents for financial support during this long endeavor.

March 12, 2007
Juhana Yrjölä

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1. INTRODUCTION

The module of a quadrilateral was introduced by H. Grötzsch in the late 1920's. It soon found many applications and become one of the most useful conformal invariants. A far reaching generalization, the module of a curve family was introduced by a Finn Lars Ahlfors and his Swedish colleague Arne Beurling in 1950.

The module of a quadrilateral is a positive real number which divides quadrilaterals into conformal equivalence classes.

2. PRELIMINARIES

The main goal of this section is to provide enough background material to fully understand the rest of this thesis. Results presented in this chapter are quite common and basic, so the reader who is well versed in mathematics can jump directly to the next chapter and return here if needed. The list of used symbols with some information is available in the appendix.

2.1. Complex numbers.

Complex numbers are a result of gradual development of our number system. This development began with the positive integers, which were used in counting. Much later it was noticed that the number system would benefit by having negative integers and zero. Gradual development lead to augmentation of rational and irrational numbers and so the real number system \mathbb{R} was born. [Spi99, p. 1]

Complex numbers were first introduced as a way to find all roots of the polynomial equations by allowing negative square roots. Later other applications were discovered in many fields like electrostatics and conformal mapping. We assume here that the reader is familiar with the basic properties and operations of complex numbers, if this is not the case, good introductory texts are [Spi99] and [Fla83], especially the first one. We use the following notation

$$(2.1) \quad z = x + yi \quad x, y \in \mathbb{R}$$

$$(2.2) \quad f(z) = u(x, y) + iv(x, y) \quad x, y \in \mathbb{R}$$

for a complex variable z and for a complex function $f(z)$, i is the imaginary unit $i = \sqrt{-1}$. The functions $u(x, y)$ and $v(x, y)$ are called conjugate functions and they are real valued.

Since complex numbers are an extended version of real numbers, many results for the real variables can be extended for complex numbers. Historically one of the most important concepts of the real analysis is the derivative. Derivative gave a birth to a completely new field of mathematics called the calculus¹. The definition

¹Historically there was a big controversy between Isaac Newton and Gottfried Leibniz about who actually invented the derivative. The quarrel started from the fact that in 1676 Leibniz was shown at least one unpublished manuscript by Newton while he visited London. It is unknown

of the derivative can be extended to complex plane quite easily, since the complex plane is in many ways similar to a Cartesian plane. The definition of the complex derivative is almost identical to the one for real functions.

2.3. Definition (Derivative for complex functions). [Spi99, p. 63]

$$(2.4) \quad f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided that $f(z)$ is single valued and the limit exists independently of the direction and manner in which $\Delta z \rightarrow 0$.

Furthermore, if the derivative $f'(z)$ exists at all points z of a region R , then $f(z)$ is said to be analytic in R and $f(z)$ is called an analytic function. There exists a nice and easy way to tell if a function $f(z)$ is analytic or not in R . If function is analytic in R , then it fulfils the Cauchy-Riemann equations and other conditions below. [Spi99, p. 63]

2.5. Theorem (Cauchy-Riemann equations). [Spi99, p. 63]

Let $f(z) = u(x, y) + iv(x, y)$, then a necessary condition for $f(z)$ to be analytic in a region R is that u and v satisfy the Cauchy-Riemann equations

$$(2.6) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If the partial derivatives are continuous in R , then the Cauchy-Riemann equations are sufficient conditions for $f(z)$ to be analytic in R . [Spi99, p. 63]. The proof can be found for example from [Spi99, pp. 72-73].

Now if the conjugate functions u and v of Theorem 2.5 have a continuous second partial derivatives in R , then we get very important set of functions called as harmonic functions. Furthermore we say that the function $f(x, y)$ is harmonic if it satisfies Laplace's equation below. [Spi99, pp. 63-64]

2.7. Definition (Laplace's equation). [Spi99, p. 63][ENC89]

Let $f(x, y)$ be a complex function in the complex plane and write

$$(2.8) \quad \nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y}.$$

Then f is said to be harmonic if and only if f satisfies the following equation

$$(2.9) \quad \nabla^2 f = 0,$$

which is known as the Laplace's equation. Also if $f(x, y) = u(x, y) + iv(x, y)$ is analytic, then u and v are harmonic.

what was in those manuscripts. Later Leibniz published the derivative first in 1684. Newton published some of his findings in 1693 and some more in 1704, almost a decade later. Currently Isaac Newton and Gottfried Leibniz are both credited for the discovery.

2.2. Curves, Jordan domains and quadrilaterals.

2.10. Definition (Metric). [AB98, p. 34]

A real valued function $d : X \times X \rightarrow \mathbb{R}$ is a metric on non-empty set X if it satisfies, for every $a, b, c \in X$, the following axioms

- M1 $d(a, b) \geq 0$ and $d(a, a) = 0$ and if $a \neq b$ then $d(a, b) > 0$.
- M2 (Symmetry) $d(a, b) = d(b, a)$
- M3 (Triangle inequality) $d(a, c) \leq d(a, b) + d(b, c)$.

The real number $d(a, b)$ is the distance from a to b . The pair (X, d) is called a metric space.

The word curve comes from Latin word *curvus* which means bent. Therefore one could define a curve as a line which has been continuously bended to different geometric shapes. The simplest curves are lines and circles. Other very old curves are conic sections which were studied by Plato who lived 430-347 B.C.[EB, V7, p. 664]

2.11. Definition (Curve). [Väi71, def. 1.1]

Curve in \mathbb{R}^2 is a continuous mapping $\gamma : [a, b] \rightarrow \mathbb{R}^2$, where $[a, b]$ is an interval in \mathbb{R} and \mathbb{R}^2 is the real plane with infinity.

Alternatively we can define curve in the complex plane as a continuous mapping $\gamma : [a, b] \rightarrow \mathbb{C}$, where \mathbb{C} is the complex plane with infinity.

For the rest of this thesis, we denote arbitrary curves by γ . Also we call set of curves a curve family and we denote a curve family by Γ . Sometimes a term path is used instead of a curve but in this work these two different terms are equal.[Väi71, def. 4.3]

2.12. Definition (Length of curve). [Väi71, def. 1.1]

Let γ be a curve, $\gamma : [a, b] \rightarrow \mathbb{R}^2$ and also let $a = t_0 \leq t_1 \leq \dots \leq t_k = b$ be a subdivision of $[a, b]$. The supremum of the sums

$$(2.13) \quad l(\gamma) = \sum_{i=1}^k |\gamma(t_i) - \gamma(t_{i-1})|$$

over all subdivisions is called the length of γ . Alternatively we can define the length of a curve as

$$(2.14) \quad l(\gamma) = \int_a^b |\gamma'(t)| dt,$$

for a curve that is differentiable.[Fla83, pp. 18-19]

We say that the curve is rectifiable if the length of a curve is finite and curve is non-rectifiable otherwise [Väi71, def. 1.1]. Furthermore we say that a curve is closed if the starting point and the end point of a curve are the same. We also say that

a curve is simple if, and only if, the curve doesn't cross itself except at the end points[Fla83, p. 16]. See Figure 1.

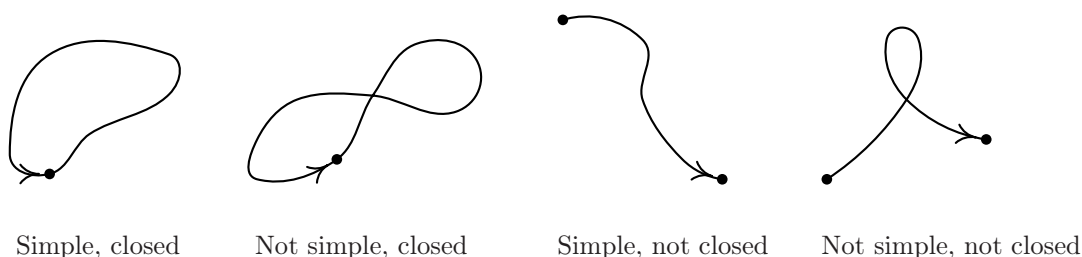


FIGURE 1. Different types of curves.

We say that an arbitrary plane set is a region. We call a region connected if it is pathwise connected, i.e., any two points can be joined by a path that is inside the region. Furthermore if a region is an open and connected set, then we call it a domain.

2.15. Definition (Simply and multiply connected domains). [Spi99, p. 93]

A connected domain D is called simply connected if any simple closed curve, which lies in D can be shrunk to a point inside D . A region which is not simply connected is called multiply connected.

Intuitively, a simply connected domain is a domain, which does not have any holes or punctures in it. Multiply connected domains have at least one hole or puncture.[Spi99, p. 94] See Figure 2.

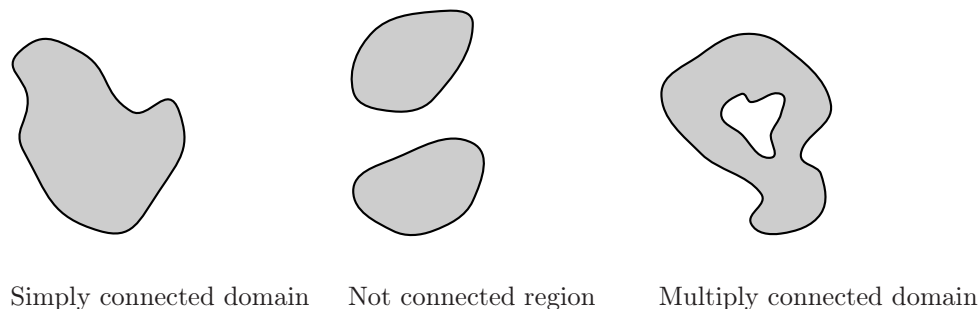


FIGURE 2. Simply and multiply connected regions.

Simple closed curves have another name in honor for French mathematician Camille Jordan, who studied complex curves.

2.16. Definition (Jordan curve). [Fla83, p. 23]

A Jordan curve is a simple closed curve.

Alternatively we can say that Jordan curve is a set which is homeomorphic to a circle. [LV73, p. 6].

Jordan noticed that every Jordan curve divides the plane into an interior domain and an exterior domain and most importantly that this obvious geometric fact required a proof. However the proof of this Theorem, which is called the Jordan curve Theorem, turned out to be quite difficult.[Fla83, p. 24] Finally in 1905 Oswald Veblen managed to prove it. In 2005 an international team of mathematicians used computer based proof system called Mizar to give a rigorous 200000-line formal proof of the Jordan curve Theorem.

2.17. Definition (Jordan domain). [Fla83, p. 24]

A Jordan domain is a bounded domain D , whose boundary is a union of a finite number of disjoint Jordan curves, where the orientations of the Jordan curves are chosen such that the interior of D is always lying to the left of the curve.

Note that our definition of a Jordan domain allows holes and punctures.

2.18. Definition (Trilateral). [Hen91, p. 428]

A trilateral is simply connected Jordan domain D with three boundary points $z_1, z_2, z_3 \in \partial D$. It is assumed that when ∂D is traversed in the positive order (i.e. the domain is on the left side) the points z_1, z_2, z_3, z_4 occur in this order. The points are called the vertices of the trilateral.

We denote a trilateral by quadruplet $T(D, z_1, z_2, z_3)$ or just $T(z_1, z_2, z_3)$ or T . Similarly we define a quadrilateral as

2.19. Definition (Quadrilateral). [LV73, p. 14]

A quadrilateral is a simply connected Jordan domain D with four distinguished boundary points z_1, z_2, z_3, z_4 of D . It is assumed that when ∂D is traversed in the positive order (i.e. the domain is on the left side) the points z_1, z_2, z_3, z_4 occur in this order. The points are called the vertices of the quadrilateral and the vertices divide the boundary curve into four parts, which are called the sides of the quadrilateral.

We use the term quadrilateral to usually mean a polygonal quadrilateral and we use the term generalized quadrilateral to mean quadrilaterals with non-polygonal sides. We denote a quadrilateral by $Q(D, z_1, z_2, z_3, z_4)$, $Q(z_1, z_2, z_3, z_4)$ or just Q .

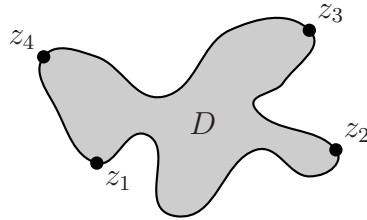


FIGURE 3. A generalized quadrilateral $Q(D, z_1, z_2, z_3, z_4)$.

Many problems of science and engineering lead to partial differential equations, which model a problem in a domain. Usually the knowledge about the behavior of the solution in the domain is not enough but we also need some knowledge about how

the solution behaves on the boundary of the domain. Usually the boundary values are introduced as extra conditions to be satisfied, and those conditions are called boundary conditions. The problem usually is to find a function which satisfies certain conditions inside the domain and some other condition on the boundary. There are two types of boundary conditions that are of great importance namely, the Dirichlet and the Neumann boundary conditions. [Spi99, p. 232]

2.20. Definition (Dirichlet and Neumann boundary conditions). [Spi99, p. 232] Let D be simply connected domain bounded by a simple closed curve γ and let f be a function, which satisfies Laplace's equation inside the domain, then

- the Dirichlet boundary condition is a condition that the function f takes prescribed values on the boundary γ .
- the Neumann boundary condition is a condition that the normal derivative $\frac{\partial f}{\partial n}$ takes prescribed values on the boundary γ .

Note that it is possible to have different types of boundary conditions on the different parts of the boundary. Also it is possible that the domain is unbounded, like the upper half plane.

Dirichlet's problem is a problem where one wants to find a function f , which satisfies Laplace's equation in the domain and also satisfies the Dirichlet boundary conditions on the boundary. Similarly Neumann's problem is a problem where one wants to find a function f , which satisfies the Laplace's equation in the domain and also satisfies the Neumann boundary conditions on the boundary. We can also talk about Dirichlet-Neumann problem, where the function f satisfies the Laplace's equation in the domain and boundary conditions consist of the Dirichlet and the Neumann conditions.[Spi99, p. 232]

Furthermore, it can be shown that solutions to the Dirichlet problem exist and are unique under suitable conditions. For instance if the boundary consists of a finite number of Jordan curves and the boundary values are continuous, then the unique solution is known to exist. Also solutions to the Neumann problem exist and are unique up to an arbitrary small additive constant. In addition, every Neumann problem can be stated in terms of an appropriately stated Dirichlet problem and the problem can contain both boundaries.[Spi99, p. 233]

2.3. Conformal maps and Riemann mapping theorem.

Gauss was the first to consider conformal maps as mathematical objects in the 1820's. However the roots of conformal maps are much older and come from the art of map making. The early mapmakers noticed that it is impossible to make distance preserving map from the spherical object earth, into the plane. Since the distances could not be preserved, mapmakers tried to preserve directions and succeeded. This lead to maps which preserve angles and the idea of the conformal map was born[Por06][Dri02, p. 4].

2.21. Definition (Conformal mapping). [Spi99, p. 201]

Let f be a mapping from z -plane into w -plane and let γ_1 and γ_2 be curves in the z -plane, which intersect at point (x_0, y_0) . The curves γ_1 and γ_2 are mapped into curves γ'_1 and γ'_2 respectively under f . Also the curves γ'_1 and γ'_2 intersect at point (u_0, v_0) . If the mapping f is such that the angle between γ_1 and γ_2 at (x_0, y_0) is equal to the angle between γ'_1 and γ'_2 at (u_0, v_0) , in both magnitude and sense, then the mapping f is conformal. See Figure 4.

It should be noted that a conformal map only preserves angles inside the domain and angles on the boundary are not necessarily preserved.

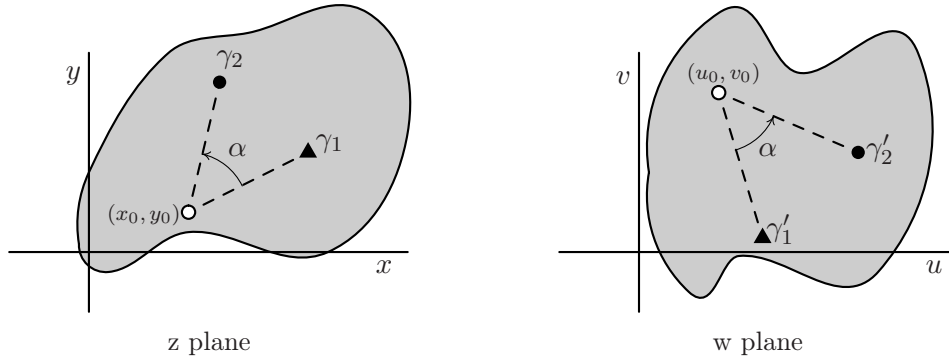


FIGURE 4. Mapping f is conformal if it preserves angles in both magnitude and sense, inside the domain. The intersection point, and the end points of the curves γ_1 and γ_2 are marked with different symbols. Note that the curves γ_1 and γ_2 need not be straight lines.

Now thanks to Gauss and many others we can use conformal maps to solve Dirichlet and Neumann problems. Usually Dirichlet and Neumann problems are quite difficult to solve. Problem domain might be complex and hard to work with. One way to overcome these difficulties is by modifying the domain into something simpler by conformal maps and solve the problem in the simplified domain. This is possible because Dirichlet and Neumann problems are conformal invariants.[Spi99, p. 232][Dri02, p. 4]

2.22. Theorem (Harmonic functions under conformal mapping). [Kre05, p. 754]

Let Φ^* be harmonic in a domain D^* in the w -plane. Suppose that $w = u + iv = f(z)$ is analytic in a domain D in the z -plane and maps D conformally onto D^* . Then the function

$$(2.23) \quad \Phi(x, y) = \Phi^*(u(x, y), v(x, y))$$

is harmonic in D .

Proof. For the proof see [Kre05, p. 754]. □

Before we can really use conformal maps to solve Dirichlet and Neumann problems we need to know which domains can be mapped to some other domains. Our hope is that these other domains might be more suited for the given problem and therefore the problem might be easier to solve in the new domain. Luckily German mathematician Bernhard Riemann gave an answer to our question, when he published the celebrated mapping theorem in 1851. Riemann's proof was considered sound at that time, however, it turned out later that the proof was not universally valid. A fact, not commonly known is that a Finn called Severin Johansson was the first person to prove the complete Riemann mapping theorem in Math. Ann. LXII (1906). Usually Constantin Carathéodory is credited for the first proof of the Riemann's mapping theorem in 1912.

2.24. Theorem (Riemann's mapping theorem). [Spi99, pp. 201-202]

Let γ be any simple closed curve in the complex plane, forming the boundary of domain D . Now there exists an analytic one to one and onto function $f(z)$ in D , which maps each point of D onto a corresponding point on the unit disk. Also this mapping maps the boundary of D into a boundary of the unit disk. The only requirement is that the domain D must be simply connected and not equal to the entire complex plane.

Now since any simply connected domain can be mapped to the unit disk conformally, then any simply connected domain can be mapped to any simply connected domain conformally. See Figure 5.

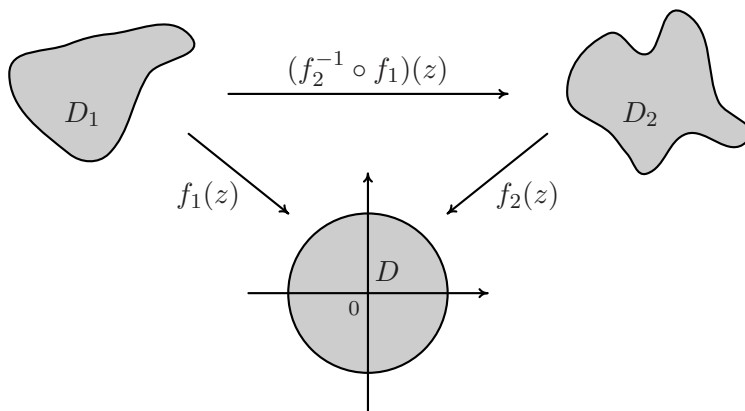


FIGURE 5. Riemann's mapping theorem and composite maps.

The only problem with the Riemann's mapping theorem is that it does not provide a formula for the mapping $f(z)$. Usually it is impossible to find this Riemann function. We can try to find out this mapping by using Möbius transformations. The advantage of this approach is in the simplicity of the Möbius transformations, however it should be noted that the Möbius transformations alone cannot yield the Riemann mapping in all cases. [Fla83, pp. 324-325]

2.25. Definition (Möbius transformation). [Fla83, p. 304]

Let a, b, c, d be given complex parameters, then the Möbius transformation with the given parameters is

$$(2.26) \quad w(z) = \frac{az + b}{cz + d} \quad ad - bc \neq 0,$$

where $z \in \mathbb{C} \cup \{\infty\}$.

Möbius transformation is sometimes called Bilinear or Fractional linear transformation.

Here we just summarize the most important properties of the Möbius transformations. For proofs and other discussion see [Fla83, pp. 306-309] and [Spi99, pp. 203, 216-217] or other conformal mapping texts.

- Möbius transformation is a conformal mapping.
- Möbius transformation is a combination of the transformations of translation, rotation, stretching and inversion.
- Möbius transformations is one-to-one and onto mapping (hence invertible).
- Möbius transformation preserves circles. Circles are mapped to circles under Möbius transformations, where straight line is circle with a infinite radius.
- Möbius transformation maps any three distinct points into any three distinct points.
- Composition of Möbius transformations is a Möbius transformation.
- The following quantity is called the cross ratio of z_1, z_2, z_3, z_4 and it is invariant under Möbius transformations.

$$(2.27) \quad \frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$$

The cross ratio is very useful in obtaining equations for Möbius transformations.

The Schwarz-Cristoffel mapping is a much more general method for obtaining Riemann's mappings and we will discuss it later.

2.28. Definition (Conformal equivalence). [Fla83, p. 326]

The domains D_1 and D_2 are said to be conformally equivalent, provided that there is a one-to-one analytic function of D_1 onto D_2 . Note that this mapping is conformal.

According to Riemann's mapping theorem all simply connected domains are conformally equivalent. We can set up additional requirements for the conformal mapping so that there does not necessarily exist a conformal map from simply connected domain to another. Most common way to set up additional requirements by introducing vertices on the boundary of the both domains and requiring that the vertices of one domain must map to vertices of the other domain.

2.29. Definition (Conformal equivalence with vertices).

Two Jordan domains D_1 and D_2 both having n -vertices labeled by v_i^1 for D_1 and v_i^2 for D_2 are called conformally equivalent if there exists a conformal map f from D_1 onto D_2 such that the continuous extension² of f to the boundary of D_1 satisfies

$$(2.30) \quad f(v_i^1) = v_i^2. \quad 1 \leq i \leq n \quad i \in \mathbb{N}$$

We can introduce up to three vertices on the boundary and still have all simply connected domains conformally equivalent. So we can conclude following for domains with three vertices.

2.31. Theorem (All trilaterals are conformally equivalent). [Hen91, Thm. 16.11a]
All trilaterals are conformally equivalent.

Proof. Let $T(z_1, z_2, z_3)$ be trilateral. Now it suffices to show that the given trilateral T is conformally equivalent to some particular trilateral $T'(z'_1, z'_2, z'_3)$. Let us fix T' to the unit disk and select three boundary points as $z'_1 = -i$, $z'_2 = 1$, $z'_3 = i$. Now let f be any conformal map from T to T' , because of Riemann's mapping theorem map f exists. Now it is possible to map the three vertices of T to the three vertices of T' by a Möbius transformation, since Möbius transformation maps circles to other circles and is characterised by three different points. So we map the vertices of T to a unit disk by the Riemann mapping and then map the unit disk to itself with a Möbius transformation, which fixes the vertices. This map can be calculated with the cross ratio. Let w be this Möbius transformation. Now the complete conformal transformation f from T to T' is $w \circ f$. Since all trilaterals can be mapped to one particular trilateral they are conformally equivalent.[Hen91, Thm. 16.11a] \square

Now if we add four vertices on the boundary we get a generalized quadrilateral. With four vertices on the boundary, a simply connected domain is not necessarily conformally equivalent anymore. This is the founding concept that this thesis work is based on. It turns out that two generalized quadrilaterals are conformally equivalent only if they have the same module. We will discuss the formal definition later.

2.4. Darboux integral.

Riemann made many other important contributions to mathematics, in addition to Riemann's mapping theorem. One of them was the concept of an integral³. We assume that the reader is familiar with the basic Riemann integration theory, where

²The Carathéodory-Osgood Theorem guarantees a continuous extension of the mapping to the boundary. It is essential that the domain is a Jordan domain for the existence of the extension. This Theorem can be found in [Hen74].[Dri02, p. 1]

³Riemann did not actually invent the concept of an integral but he did generalize the finds by Archimedes and Eudoxus. Archimedes used the method of Eudoxus to compute the area of a disk. Interestingly, the value of the limit of the areas of the inscribed(or circumscribed) polygons that were employed by Archimedes in his area computations, was also called by him the integral ($\tau\acute{o}\pi\acute{\alpha}\nu$).[AB98, p. 180]

the Riemann integral was defined with Riemann sums. Since the work of Riemann, many other definitions for the integral have surfaced. One of these definitions was given by a French mathematician Gaston Darboux in 1875. It has the advantage that it is slightly simpler to define than the Riemann one.

2.32. Definition (Darboux Sums). [ENC89]

Let a real function f be defined and bounded on a segment $[a, b]$, let $\tau = \{x_i\}_{i=0}^k$ be a decomposition of $[a, b]$:

$$(2.33) \quad a = x_0 < x_1 < \cdots < x_k = b,$$

and set

$$(2.34) \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x), \quad M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x),$$

$$(2.35) \quad \Delta x_i = x_i - x_{i-1}, \quad i = 1, \dots, k.$$

The sums

$$(2.36) \quad s_\tau = \sum_{i=1}^k m_i \Delta x_i \quad \text{and} \quad S_\tau = \sum_{i=1}^k M_i \Delta x_i$$

are known, respectively, as the lower and upper Darboux sums.

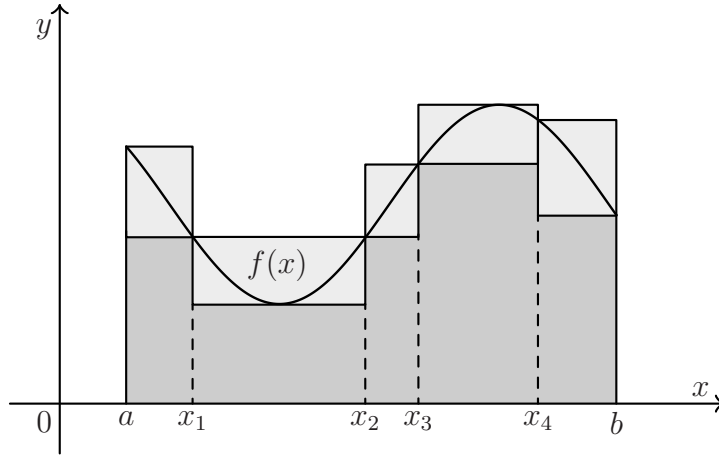


FIGURE 6. The geometric meaning of the lower and upper Darboux sums. The lower Darboux sum(s_τ) is the dark gray area and the upper Darboux sum(S_τ) is the dark gray area plus the light gray area. It is obvious from the Figure that $s_\tau \leq S_\tau$.

The limits of the above Darboux sums are called Darboux integrals.

2.37. Definition (Darboux integrals). [ENC89]

The numbers

$$(2.38) \quad I_* = \sup_{\tau} s_\tau, \quad I^* = \inf_{\tau} S_\tau$$

are called, respectively, the lower and the upper Darboux integrals of f . They are the limits of the lower and upper Darboux sums:

$$(2.39) \quad I_* = \lim_{\delta_\tau \rightarrow 0} s_\tau, \quad I^* = \lim_{\delta_\tau \rightarrow 0} S_\tau$$

Where

$$(2.40) \quad \delta_\tau = \max_{i=1, \dots, k} \Delta x_i.$$

δ_τ is called the fineness (mess) of the decomposition.

2.41. Theorem (Riemann and Darboux integrable functions). [ENC89]

The necessary and sufficient condition for a function f to be Riemann integrable on the segment $[a, b]$ is

$$(2.42) \quad I_* = I^*.$$

If the above condition is met, then the value of the lower and the upper Darboux integrals becomes identical with the Riemann integral

$$(2.43) \quad \int_a^b f(x) dx.$$

For the proof of the Theorem see [AB98, pp. 177-187].

2.5. Elliptic integrals.

Many interesting and useful integrals cannot be expressed in terms of elementary functions, such as the length of the rectified lemniscate, capacitance of an ellipsoid and map of upper half-plane to a rectangle. These integral functions, known as Elliptic integrals, were exhaustively studied in 1797-1829 by Gauss, Legendre, Abel and Jacobi. Furthermore they showed that every elliptic integral can be reduced to one of the three normal forms. These normal forms are called elliptic integral of the first, the second and the third kind [MM99, pp. 54-65]. Here we are only interested in elliptic integrals of the first kind because they map upper half-plane to a rectangle.

2.44. Definition (The elliptic integral of the first kind). [MM99, pp. 55-57] [SL99, p. 195]

$$(2.45) \quad F(k, \phi) = \int_0^{\sin \phi} \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}},$$

Where $0 < k < 1$, k is called the elliptic modulus and ϕ is called the amplitude. Also we will denote the complementary elliptic modulus by k' , where $k' = \sqrt{1-k^2}$.

If we are only interested in the borders of the image rectangle we can use simplified version of the elliptic integral $F(k, \phi)$. This simplified integral is called the complete elliptic integral of the first kind.

2.46. **Definition** (The complete elliptic integral of the first kind). [MM99, pp. 55-57]

$$(2.47) \quad \mathcal{K} = F(k, \frac{\pi}{2}) = \int_0^1 \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}},$$

Where $0 < k < 1$. Also we will denote the complete complementary integral of the first kind by $\mathcal{K}'(k)$ or just \mathcal{K}' , where \mathcal{K}' is the complete elliptic integral of the first kind formed with the complementary elliptic modulus, $\mathcal{K}' = \mathcal{K}(k')$.

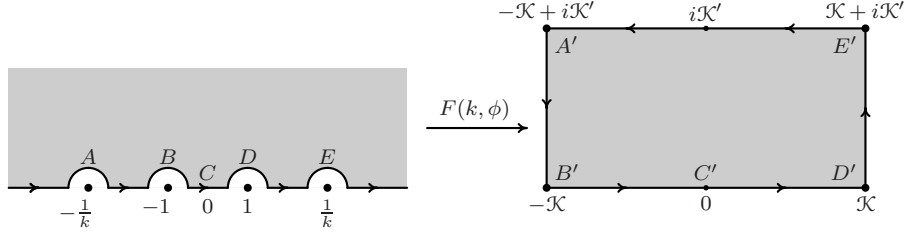


FIGURE 7. Map of the upper half-plane by elliptic integral of the first kind. Note that map has branch points at $-\frac{1}{k}$, -1 , 1 and $\frac{1}{k}$. These branch points can be avoided by traversing along infinitesimal semi-circles. Primed letters A', B', C', D', E' are the images of unprimed letters.

Above elliptic integrals can be obtained by using the Schwarz-Christoffel mapping. Usually a numerical method must be used to evaluate elliptic integrals. Numerical methods for the elliptic integrals of the first kind are discussed in the appendix.

2.6. Numerical methods and accuracy.

Numerical method is computer aided way of calculating something. Usually a numerical method is utilized when we cannot solve problem with just pencil and paper. Modern high speed computers have made numerical methods more and more common but one should be careful of using a numerical method blindly because there are several things that can go wrong and cause errors. The most important issues that can go wrong with a numerical method are

- Accuracy
- Stability
- Efficiency

Most importantly a numerical method must produce accurate results. Only very few numerical methods produce exact results. Computers cannot store numbers with infinite precision, since they have limited amount of memory. Limited memory especially affects floating point numbers, since integers are easier to represent accurately. Because of the limited accuracy of floating-point numbers there exists a number depending on the system which can be added to the floating-point number 1.0 and the result is not affected. This number is called machine epsilon ϵ_m (a typical

size is about 3×10^{-16} for modern computers). Virtually every arithmetic operation with floating numbers could cause an additional error of at least ϵ_m . This type of error is called roundoff error.[NRC, pp. 29-30]

A programmer can cause inaccuracies as well. A very common source of errors are for example "discrete" approximations to some continuous quantity or errors caused by truncation to finite accuracy.[NRC, p. 30]

Sometimes a method can work theoretically well on paper but even small amount of roundoff error causes the method go wrong. These kinds of methods are called unstable.[NRC, p. 30]

Last but not least a numerical method must be sufficiently fast and effective to compute. The most accurate method, which cannot be used because of time constraints is useless.

The most important thing to remember is that numerical methods are not 100% accurate.

3. THE MODULE OF A QUADRILATERAL

The module of a quadrilateral is positive real number which divides generalized quadrilaterals into conformal equivalence classes. Historically words module and modulus have been used interchangeably and we have chosen to use the first one. The only difference between them is that the word module is an English word with plural modules and the modulus is a Latin word with plural moduli.

There are many equal definitions for the module of a quadrilateral as well. In this section we investigate how the module of the quadrilateral can be defined by the module of a curve family or by the Dirichlet-Neumann problem. The module of the quadrilateral can also be defined by using harmonic measure but it is beyond the scope of this text. The harmonic measure approach attaches the module concept to stochastic processes and it is possible to stochastically to solve the Dirichlet-Neumann problem by Brownian random walk. More discussion about the harmonic measure see for example [Küh05] and [Ian03].

3.1. The module of a curve family.

The notion of the module of a curve family was first introduced by Lars Ahlfors and Arne Beurling in 1950, in their publication "Conformal invariants and function theoretic null-sets"[AhBe50]. Although Ahlfors and Beurling used the name extremal length instead of module of a curve family at the time of the publication, extremal length is closely related to the current definition of the module of a curve family. Extremal length is the reciprocal of the module of a curve family. Thanks

to the work of Ahlfors and Beurling, active development of the theory began and already in 1957 Bengt Fuglede generalized their notion and gave it a measure-theoretic interpretation.[Väi71, p. 20][Vas02, p. 1]

Before we can give the formal definition of module of a curve family we need to define some auxiliary concepts.

Let $\rho(z)$ be non-negative, real valued, continuous⁴ and integrable function in some domain D of \mathbb{C} . Now this function $\rho(z)$ defines a differential metric ρ on D by $\rho := \rho(z)|dz|$.

3.1. Definition (ρ -length). [Vas02, p. 8]

Let D be a domain in \mathbb{C} and let γ be a curve in D , then the lower Darboux integral⁵

$$(3.2) \quad l_\rho(\gamma) = \int_\gamma \rho(z)|dz|$$

is called the ρ -length of γ . Note that if $\rho(z) = 1$ almost everywhere and γ is rectifiable, then $l_\rho(\gamma)$ is Euclidean length of $\gamma \subset D$. Let Γ be a family of curves γ in D , then the next quantity

$$(3.3) \quad L_\rho(\Gamma) = \inf_{\gamma \in \Gamma} l_\rho(\gamma)$$

is called the ρ -length of the curve family Γ .

Reason for the above notation for $L_\rho(\Gamma)$ becomes clear soon, when the definition of the module of the curve family is given.

3.4. Definition (ρ -area). [Vas02, p. 8]

Let D be a domain in \mathbb{C} and let γ be a curve in D , then the integral

$$(3.5) \quad A_\rho(D) = \iint_D \rho^2(z) d\sigma_z, \quad d\sigma_z = dx \cdot dy$$

is called the ρ -area of D .

3.6. Definition (The module of the curve family). [Vas02, p. 8]

Let D be a domain in \mathbb{C} and let Γ be a curve family in D , then the quantity

$$(3.7) \quad m(D, \Gamma) = \inf_\rho \frac{A_\rho(D)}{L_\rho^2(\Gamma)}$$

⁴Here the continuous requirement is a simplification that we do here, since we have not discussed measure theory in the background chapter. The whole theory must be applied to Borel functions $\rho(z)$, which satisfy other conditions given above. However for the module of a quadrilateral computations continuous functions are enough[Väi71, p. 20]. Note that all continuous functions are Borel functions but not all Borel functions are continuous.

⁵Here the lower Darboux integral is chosen so that the integral always exists even if the curve γ is non-rectifiable, however the curve must be locally rectifiable. This discussion is purely academic since in the module of a quadrilateral computations we need to restrict to rectifiable curves anyway. See Theorem 3.13.

is called the module of the curve family Γ in D where the infimum is taken over all metrics ρ in D . $A_\rho(D)$ is the ρ -area of D and $L_\rho(\Gamma)$ is the ρ -length of the curve family Γ .

Above definition 3.6 looks quite formidable but if we strip it down we get essentially the following formula

3.8. Definition (Module formula). [CFP96, p. 1]

$$(3.9) \quad m_\rho = \frac{A_\rho}{L_\rho^2},$$

where the module m_ρ is a ratio of certain area A_ρ and square of some length L_ρ . the ρ is a metric chosen so that the ratio is minimal. The metric ρ tells us how to compute the area and the length. It should be emphasized that the metric ρ does not have to be the Euclidean metric and with Euclidean area and length above formula might produce wrong results. Later we see inequality version of this very same formula that is always valid.

Luckily it is possible to simplify Definition 3.6 by reducing the set of all the possible metrics. The set of possible metrics is reduced so that term $L_\rho^2(\Gamma)$ could be removed from the Definition 3.6. Metrics that remain after the reduction are called admissible metrics.

3.10. Definition (Admissible metric). [Vas02, p. 8]

Let ρ be a metric in D , then metric ρ is admissible if $l_\rho(\gamma) \geq 1$. Also we denote by $P(\Gamma)$ the set of all admissible metrics in D .

3.11. Theorem (Simplified module of the curve family). [Vas02, p. 8]

Let D be a domain in \mathbb{C} and let Γ be curve family in D . Let $P(\Gamma)$ be set of all admissible metrics in D , then we can define the module of the curve family as

$$(3.12) \quad m(D, \Gamma) = \begin{cases} \inf_{\rho \in P(\Gamma)} A_\rho(D) & \text{if } P(\Gamma) \neq \emptyset \\ \infty & \text{if } P(\Gamma) = \emptyset \end{cases}$$

where the infimum is taken over all metrics ρ in $P(\Gamma)$.

A metric ρ^* is called extremal if $m(D, \Gamma) = A_{\rho^*}(D)$. For the rest of this thesis we will denote extremal metric by ρ^* . Also the extremal length of Γ is $1/m(D, \Gamma)$. [Vas02, p. 8][Väi71, p. 16]

We use mainly the above simplified version of the module of a curve family, when we calculate modules.

3.13. Theorem (Non-rectifiable curves have module zero). [Väi71, Cor. 6.11]

The family of all non-rectifiable curves in \mathbb{R}^n has module zero.

Proof. For the proof see [Väi71, Thm. 6.9] □

Next we have very important result.

3.14. Theorem (Conformal invariance of the module of a curve family). [Vas02, Thm. 2.1.1]

Let Γ be a family of curves in a domain $D \in \bar{\mathbb{C}}$, and let $w = f(z)$ be a conformal map of D onto $D' \in \bar{\mathbb{C}}$. If $\Gamma' = f(\Gamma)$, then

$$(3.15) \quad m(D, \Gamma) = m(D', \Gamma').$$

Proof. Let P' be the family of all admissible metrics for Γ' and let P be the family of admissible metrics for Γ . Let us set $\rho(z)|dz| = \tilde{\rho}(f(z))|f'(z)||dz|$ for $\tilde{\rho} \in P'$. Since

$$(3.16) \quad \int_{\gamma \in \Gamma} \rho(z)|dz| = \int_{\gamma} \tilde{\rho}(f(z))|f'(z)||dz| = \int_{f(\gamma) \in \Gamma'} \tilde{\rho}(w)|dw| \geq 1,$$

for any $\gamma \in \Gamma$, we have $\rho \in P$. By change of variables under a conformal map in 3.5 we have $A_\rho(D) = A_{\tilde{\rho}}(D')$. Hence,

$$(3.17) \quad m(D, \Gamma) = \inf_P A_\rho(D) \leq \inf_{\tilde{\rho} \in P'} A_{\tilde{\rho}}(D') = m(D', \Gamma').$$

Considering the inverse map $z = f^{-1}(w)$ we obtain the reverse inequality and then the proof is finished. [Vas02, p. 9] \square

Next results are good to keep in mind when trying to calculate the module of the curve family.

3.18. Theorem (Properties of the extremal metric). [Vas02, Thms. 2.1.2-2.1.3]

(i) (*Uniqueness of the extremal metric*):

Let ρ_1^* and ρ_2^* be two extremal metrics for the module $m(D, \Gamma)$, then $\rho^* = \rho_1^* = \rho_2^*$ almost everywhere.

(ii) (ρ^* -length of the curve family Γ):

$$L_{\rho^*}(\Gamma) = 1$$

(iii) (*Monotonicity*):

If $\Gamma_1 \subset \Gamma_2$ in D , then $m(D, \Gamma_1) \leq m(D, \Gamma_2)$.

Proof. (i) Since ρ_1 and ρ_2 are extremal metrics for the module $m(D, \Gamma)$, they are admissible. Then $\frac{1}{2}(\rho_1(z) + \rho_2(z))|dz|$ is an admissible metric, and

$$(3.19) \quad \iint_D \left(\frac{\rho_1(z) + \rho_2(z)}{2} \right)^2 d\sigma_z \geq m(D, \Gamma).$$

We have the chain of inequalities

$$(3.20) \quad 0 \leq \iint_D \left(\frac{\rho_1(z) - \rho_2(z)}{2} \right)^2 d\sigma_z$$

$$(3.21) \quad = \iint_D \left(\frac{\rho_1^2(z) + \rho_2^2(z)}{2} \right) d\sigma_z - \iint_D \left(\frac{\rho_1(z) + \rho_2(z)}{2} \right)^2 d\sigma_z$$

$$(3.22) \quad = m(D, \Gamma) - \iint_D \left(\frac{\rho_1(z) + \rho_2(z)}{2} \right)^2 d\sigma_z \leq 0,$$

which is valid only if $\rho_1(z) = \rho_2(z)$ almost everywhere. [Vas02, p. 9]

(ii) We denote by $\rho^*(z)|dz|$ this extremal metric. Suppose that $L_{\rho^*}(\Gamma) = c > 1$. Then the metric $\frac{1}{c}\rho^*(z)|dz|$ is admissible and

$$(3.23) \quad m(D, \Gamma) \leq \frac{1}{c^2} \iint_D (\rho^*(z))^2 d\sigma_z = \frac{1}{c^2} m(D, \Gamma) < m(D, \Gamma)$$

This is a contradiction, proving (ii). [Vas02, p. 9]

(iii) This follows from the inequality $L_\rho(\Gamma_1) \geq L_\rho(\Gamma_2)$. [Vas02, p. 10] □

Now it is time to investigate the connection between the module of a curve family and the module of a quadrilateral. These concepts are not completely the same, since the module of a curve family is a more general concept.

3.24. Definition (Module of a quadrilateral). [Küh05, p. 107]

Let Q be a quadrilateral with four vertices z_1, z_2, z_3 and z_4 . Let τ_1 and τ_2 be two opposite sides of the quadrilateral and let Γ_1 be a curve family of all rectifiable curves that connect the two opposite sides τ_1 and τ_2 of Q . Then the module of a quadrilateral $Q(z_1, z_2, z_3, z_4)$ with τ_1 and τ_2 as opposite sides is the module of the curve family Γ_1 .

We denote the module of a quadrilateral with capital M , in contrast to the lower case m , which denotes the module of a curve family. Also note that since the module of a curve family is conformal invariant by Theorem 3.14 then the module of a quadrilateral is also a conformal invariant.

If we replace the opposite sides τ_1 and τ_2 of Q by the other set of opposite sides then module of the quadrilateral Q is equal to reciprocal module $1/M(Q)$. [Küh05, p. 101] If we do not take all the curves that connect opposite sides of Q then we get following inequality

$$(3.25) \quad m(Q, \Gamma) \leq M(Q),$$

where Γ is the curve family that joins the opposite sides. Equality holds if we take so called module line curves with the extremal metric⁶. There is a simple geometric interpretation of the module lines. If we have a conformal map that maps a generalized quadrilateral onto a rectangle and the vertices of the quadrilateral are mapped to corners of the rectangle, then the module lines are inverse images of the straight lines that connect the two opposing sides of the rectangle. See Figure 8. [Küh05, pp. 101,109]

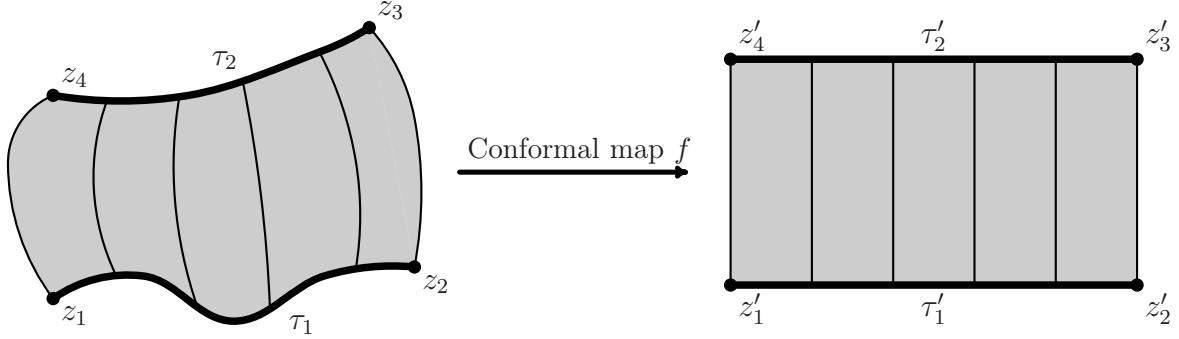


FIGURE 8. Module lines of a quadrilateral.

If we do not take all the possible metrics ρ , then we get the following inequality

$$(3.26) \quad m(Q, \Gamma) \geq M(Q),$$

where equality holds if the extremal metric ρ^* is included. Note that if we reduce curves and metrics at the same time it is very difficult to say what happens.

Generally speaking calculating the module of a quadrilateral with the module of a curve family definition is quite difficult as the next example will show.

3.27. **Example** (Module of a rectangle⁷).

Let Q be a rectangle of length l and height h . This rectangle is shown in Figure 9. Let Γ be the family of all rectifiable curves in Q that connect two opposite horizontal sides of Q , then $M(Q) = \frac{l}{h}$

⁶Note that module lines depend on the extremal metric and with wrong metric length of the module lines might be wrong.

⁷This example uses construction from Example 2.1.3 of [Vas02, pp. 10-11], however we have proven slightly more general result.

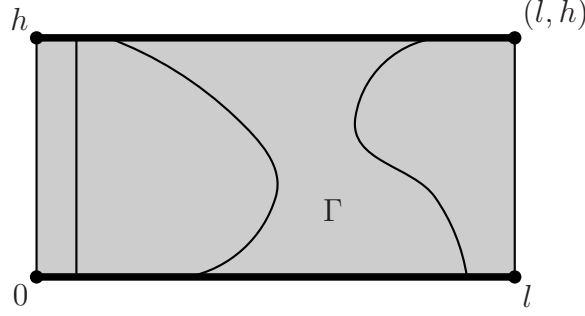


FIGURE 9. A rectangle with some curves. These curves and many others form a curve family Γ on a rectangle. Opposite sides are marked as thick lines.

Proof. The idea of the proof is to pick up some metric ρ and show that is extremal one. Now let us pick Euclidean metric divided by a scaling constant h . The metric $\rho = \frac{|dz|}{h}$ is admissible, and therefore the set P of admissible metrics is non-empty. Now let us calculate the ρ -area of the rectangle Q .

$$(3.28) \quad A_\rho(Q) = \int_0^h \int_0^l \frac{1}{h^2} dx dy = \frac{1}{h^2} lh = \frac{l}{h}$$

Now since we restricted the set of metrics have $M(Q) \leq m(Q, \Gamma) \leq \frac{l}{h}$

Now, let ρ be any admissible metric from P . Then

$$(3.29) \quad 1 \leq \int_0^h \rho(x + yi) dy$$

and if we integrate the previous result again we get

$$(3.30) \quad l \leq \int_0^l \left(\int_0^h \rho(x + yi) dy \right) dx = \iint_D \rho(z) dx dy.$$

Now let us form the following chain of inequalities

$$(3.31) \quad 0 \leq \iint_D \left(\frac{1}{h} - \rho(z) \right)^2 dx dy$$

$$(3.32) \quad = \iint_D \frac{1}{h^2} dx dy - \frac{2}{h} \iint_D \rho(z) dx dy + \iint_D \rho^2(z) dx dy$$

$$(3.33) \quad \leq \frac{l}{h} - 2\frac{l}{h} + \iint_D \rho^2(z) dx dy = -\frac{l}{h} + \iint_D \rho^2(z) dx dy$$

The chain leads to the following inequality

$$(3.34) \quad \iint_D \rho^2(z) dx dy \geq \frac{l}{h}.$$

for any admissible ρ and, taking the infimum over all ρ , we have $M(Q) = m(Q, \Gamma) \geq \frac{l}{h}$. Now since $\frac{l}{h} \leq M(Q) \leq \frac{l}{h}$ we see that

$$(3.35) \quad M(Q) = \frac{l}{h}$$

□

Note that the module of a quadrilateral for a rectangle is the aspect ratio of the rectangle. Also note that if we switch the opposite sides of Q , then we get the reciprocal module $\frac{h}{l}$

Since the exact calculation of the module of the quadrilateral is very difficult and tiresome, inequalities have a major role in getting estimates for the modules.

3.36. Theorem (Rengel's inequality). [LV73, p. 22]

The module of a quadrilateral Q satisfies the double inequality

$$(3.37) \quad \frac{(L_\rho(\Gamma'))^2}{A_\rho} \leq M(Q) \leq \frac{A_\rho}{(L_\rho(\Gamma))^2}.$$

Where Γ' and Γ are curve families that connect the opposite sides of a quadrilateral Q . Equality holds in both cases when and only when Q is a rectangle.

Proof. For the proof see [LV73, pp. 22-23]

□

Note that the Rengel's inequality is inequality version of the formula (3.9)⁸.

3.2. The Dirichlet-Neumann problem and the module of a quadrilateral.

There is another definition for the module of a quadrilateral, which is based on Laplace's equation on a rectangle. This formulation actually gives us three different definitions. Namely definitions that are based on Laplace's equation, geometric shape of the quadrilateral and capacitance of a condenser. All these definitions are very closely related as we are about to see.

First we take a look at the Dirichlet-Neumann formulation for the module of a quadrilateral. It should be emphasized that the Dirichlet-integral given below is indeed a conformal invariant, although we do not prove it here. The proof was given by Loewner in 1959, see [Loe59].

3.38. Theorem (Dirichlet-Neumann definition). [Hen91, pp.431-432]

Let Q be a quadrilateral with vertices a, b, c, d and edges $(a,b), (b,c), (c,d), (d,a)$. Now the module of a quadrilateral can be determined by solving the following

⁸It is also possible to estimate the module of a quadrilateral with two Euclidean lengths and no information about the area. See [LV73, p. 23].

Dirichlet-Neumann boundary value problem. See Figure 10.

$$(3.39) \quad \nabla^2 \phi = 0 \quad \text{in } Q$$

$$(3.40) \quad \phi = 0 \quad \text{on } (a, b),$$

$$(3.41) \quad \phi = 1 \quad \text{on } (c, d),$$

$$(3.42) \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } (b, c) \text{ and } (d, a),$$

where $\partial/\partial n$ denotes differentiation in the direction of the exterior normal and ϕ is the solution of the Dirichlet-Neumann problem. After we have solved the function ϕ , then the module of the quadrilateral Q is given by

$$(3.43) \quad \frac{1}{M(Q)} = \int_c^d \frac{\partial \phi}{\partial n} ds$$

Generally speaking the calculation of the integral (3.43) in the domain Q is difficult. However we can map the domain Q into a rectangle Q' with a conformal map and calculate the integral there. See Figure 10. [Hen91, pp.431-432]

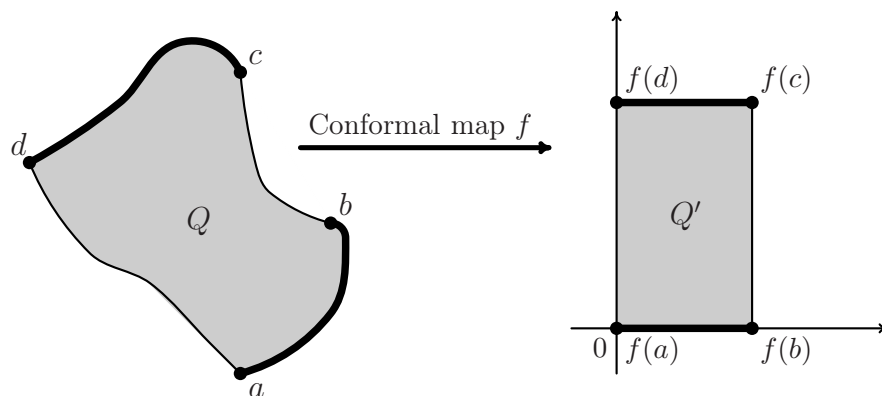


FIGURE 10. Boundary value problem. The Dirichlet boundary conditions are marked as thick lines. The Neumann boundary conditions are thin lines.

Furthermore the integral (3.43) has an interesting physical interpretation. If the quadrilateral Q is considered to be a thin sheet of metal with electrical resistance 1 and sides (a, b) and (c, d) are kept at potentials 0 and 1 respectively, while other sides are insulated then the integral (3.43) is the total amount of current that passes through the sheet of metal. If we were to construct such a sheet of metal then its capacitance would be given with the following formula

3.44. Theorem (Capacitance of a condenser). [Küh05, pp. 102-103]

$$(3.45) \quad C = \frac{1}{4\pi} \int_a^b \frac{\partial \phi}{\partial n} ds.$$

Also if we do know the module of any quadrilateral, then we can compute the capacitance of that quadrilateral with following module-capacitance formula.

3.46. Theorem (Module-capacitance formula⁹). [Gai79, p. 236]

Let $M(Q)$ be the module of the quadrilateral Q with vertices a, b, c and d . Then the capacitance between the sides (a, b) and (c, d) is given by

$$(3.47) \quad C = \frac{1}{\pi^2} \sum_{n=odd} \frac{1}{n \sinh(n\pi M(Q))}$$

Since the module of a quadrilateral is invariant under conformal mappings we can simplify the Dirichlet-Neumann domain by a conformal mapping. Because we already know how to compute the module of a rectangle (See Example 3.27), then a conformal map to a rectangle gives us the following geometric definition for the module of a quadrilateral.

3.48. Definition (Geometric definition). [Küh05, p. 101]

Let Q be a quadrilateral with vertices a, b, c and d . Let f be a conformal map that maps Q into a rectangle and vertices a, b, c and d map into corners of that rectangle so that vertex a maps to zero and vertex b maps on a real axis. See Figure 10. Then the module of Q is given by

$$(3.49) \quad M(Q) = \frac{\operatorname{Im} f(c)}{\operatorname{Re} f(b)} = \frac{\text{height}}{\text{length}}$$

where $\operatorname{Im} f(c)$ is the imaginary part of $f(c)$ and $\operatorname{Re} f(b)$ is the real part of $f(b)$.

The only problem with this geometric definition is that it requires a construction of a conformal map that maps a quadrilateral into a rectangle. In the next section we are going to look at how this map could be constructed for polygonal quadrilaterals.

4. THE SCHWARZ-CHRISTOFFEL MAPPING

In 1851 the German mathematician Bernhard Riemann presented in his doctoral dissertation the famous Riemann's mapping theorem that guarantees that any simply connected domain in the complex plane can be mapped to some other simply connected domain by a conformal mapping. However Riemann's theorem is an existence Theorem only and it does not give the required mapping function. Soon after the Riemann's discovery, Elwin Christoffel(1867) and Hermann Schwarz (1869) independently discovered the Riemann mapping function in a case where the range of the mapping domain is a polygon. This mapping is called the Schwarz-Christoffel mapping.[Fla83][Dri02]

⁹Matlab implementation of this algorithm is given in the Schwarz-Christoffel toolbox section.

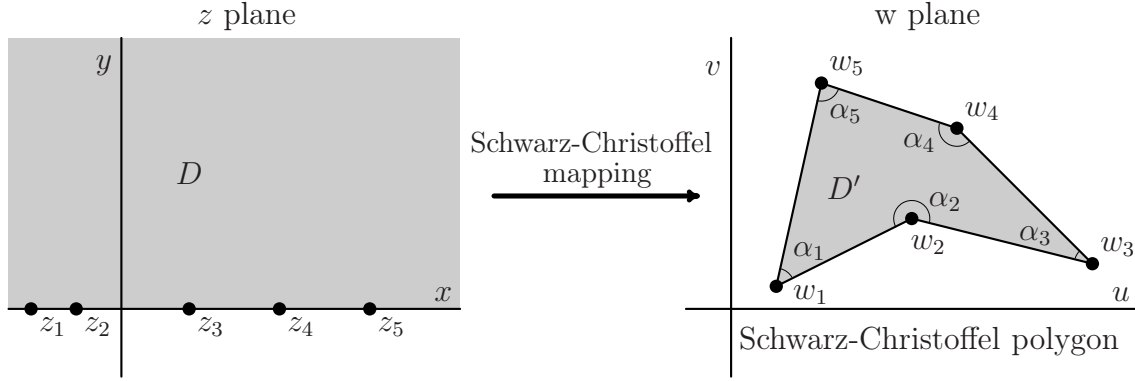


FIGURE 11. Schwarz-Christoffel mapping. Here the prevertices are z_1, z_2, z_3, z_4 and z_5 . The images of the prevertices are respectively w_1, w_2, w_3, w_4 and w_5 .

Consider a polygon in Figure 11. The polygon has vertices w_1, w_2, w_3, w_4 and w_5 with corresponding interior angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 . In the above figure each highlighted point in the z plane maps to specific polygon vertex in the w plane. These highlighted points are called prevertices and the image of the prevertex under Schwarz-Christoffel mapping is always a vertex in the Schwarz-Christoffel polygon. Also note that the number of prevertices determines the number of corners in the Schwarz-Christoffel polygon¹⁰. [Dri02]

4.1. Theorem (Schwarz-Christoffel Mapping for the upper half plane). [Spi99, p. 204]

A transformation which maps the upper half plane D to the interior of arbitrary polygon D' is called Schwarz-Christoffel mapping. This same mapping also maps the real axis to the boundary of the polygon¹¹ D' . The Schwarz-Christoffel mapping for the upper half plane is given by

$$(4.2) \quad w(z) = B + C \int^z (x - z_1)^{\frac{\alpha_1}{\pi} - 1} (x - z_2)^{\frac{\alpha_2}{\pi} - 1} \cdots (x - z_n)^{\frac{\alpha_n}{\pi} - 1} dx$$

Where B and C are some complex constants that need to be determined. Index n is the number of prevertices. The points w_n and angles α_n determine the polygon as above in Figure 11. The lower integration limit is left unspecified, since it affects only the value of B . [Dri02, p. 3]

Proof. We do not actually give rigorous proof to this Theorem. Instead we do give some insight how the above formula is formed. A rigorous proof can be found for example from [Spi99, p. 223] or from [Dri02, pp. 10-11].

¹⁰It is possible to map a prevertex to polygon corner which has an angle π or 2π . [Dri02]

¹¹The Carathéodory-Osgood Theorem guarantees a continuous extension of the mapping to the boundary. This Theorem can be found in [Hen74]. [Dri02, p. 1]. It should be also noted that, a conformal map only preserves angles in the interior domain and not on the boundary. This allows the Schwarz-Christoffel mapping to map the real axis into boundary of any arbitrary polygon.

First we differentiate equation (4.2) in order to get rid of the integral sign and the constant B . We obtain

$$(4.3) \quad \frac{dw}{dx} = C(x - z_1)^{\frac{\alpha_1}{\pi}-1}(x - z_2)^{\frac{\alpha_2}{\pi}-1} \cdots (x - z_n)^{\frac{\alpha_n}{\pi}-1}.$$

Now observe that from 4.3 we have

$$(4.4) \quad \begin{aligned} \arg dw &= \arg dx + \arg C + \left(\frac{\alpha_1}{\pi} - 1\right) \arg(x - z_1) \\ &+ \left(\frac{\alpha_2}{\pi} - 1\right) \arg(x - z_2) + \cdots + \left(\frac{\alpha_n}{\pi} - 1\right) \arg(x - z_n) \end{aligned}$$

Now as x moves along the real axis from left toward z_1 , we can assume that w moves along a side of the polygon toward w_1 . When x moves from the left of z_1 to the right of z_1 , then the $\theta_1 = \arg(x - z_1)$ changes from π to 0 while all other terms in (4.4) stay constant. This means that $\arg dw$ must decrease, since the right side of the equation (4.4) decreased. The amount of decrease is $\left(\frac{\alpha_1}{\pi} - 1\right) \arg(x - z_1) = \left(\frac{\alpha_1}{\pi} - 1\right) \pi = \alpha_1 - \pi$. Alternatively we can talk about an increase of $\pi - \alpha_1$. See Figure 12.

This increase has the effect that the w turns through the angle $\pi - \alpha_1$, and after the increase w moves along the side (w_1, w_2) of the Schwarz-Christoffel polygon. This same thing happens again when the x moves through z_2 and w turns $\pi - \alpha_2$. By continuing this process, we notice how the upper half plane is mapped into interior of the polygon.[Spi99, pp. 218-219] \square

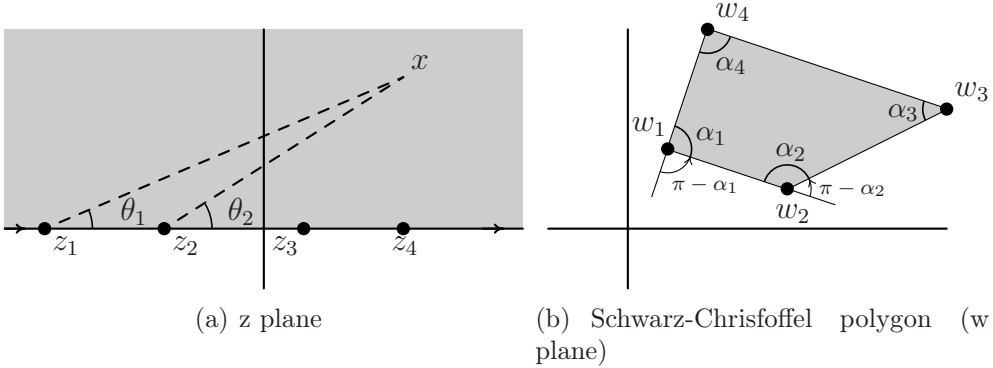


FIGURE 12. Creation of the Schwarz-Christoffel transformation.

Note that in order to use the Schwarz-Christoffel mapping we need to determine all the prevertices and constants in advance. This problem of determining prevertices and constants is called the Schwarz-Christoffel parameter problem. Parameter problem is one of the most difficult tasks in the Schwarz-Christoffel mapping. Generally speaking, the determination of the constants B and C are relatively simple

after the prevertices have been determined. Usually a numerical scheme must be used to determine the prevertices.[Dri02, pp. 23-40]

4.1. Numerical implementation of the Schwarz-Christoffel mapping.

A Schwarz-Christoffel mapping consists of many different components. These components can be seen from Figure 13.

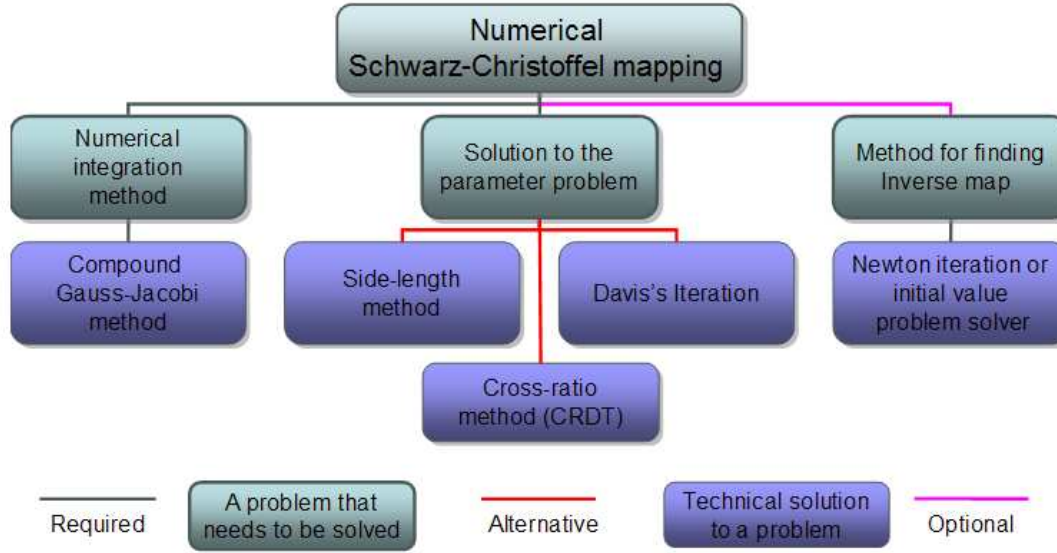


FIGURE 13. The basic components of the numerical Schwarz-Christoffel mapping

The three main components are a numerical integration method, a parameter problem solver and a method for inverting the map. The solution to the parameter problem is the most difficult one. Currently there are three well known methods to accomplish this. One of the simplest methods for the solution of the parameter problem is called Davis's iteration.

4.5. **Theorem** (Davis's iteration for a half plane). [How90, p. 62]

$$(4.6) \quad (z_{i+1} - z_j)_{new} = k(z_{i+1} - z_i) \cdot \frac{|w'_{i+1} - w'_i|}{|w_{i+1} - w_i|}$$

Where z' and w' are the unknown correct prevertices and the desired corners, respectively for the polygon. Variables z and w are approximations based on incorrect prevertex positions, so that z and w are the iterated values of prevertices and the desired corners, respectively. Value of the variable k depends on the problem setup and trial and error. It is even possible to ignore variable k completely at cost of impaired performance.

Davis's iteration is computationally cheap (compared to other methods) and simple to implement. Because of its cheap computation time, it has been used to solve the parameter problem for polygons with thousand or more vertices[BT03]. However H. Howell pointed out that Davis's iteration is not the best way to solve the parameter problem for the following reasons:

- Parameters for the Davis's iteration change case-by-case basis(namely k). Determining the right ones for the problem at hand is often impractical.
- Many Schwarz-Christoffel problems have additional conditions to be satisfied. For example if the conformal module is known and one polygonal side length is left unspecified. To accomplish this with Davis's iteration is hard.
- Convergence rate near the solution is only linear.
- The most critical fact is that Davis's iteration might not converge at all.

For more details see[How90, p. 62-64].

The second method for the solution of the parameter problem is called Side-length method. This method was introduced in 1980 by Nick Trefethen and variations of this method remain the mainstay of the numerical Schwarz-Christoffel mapping[Dri02, p. 23]. Side-length method is significantly more difficult than Davis's iteration but is more powerful.

4.7. Theorem (Side-length method). [Dri02, p. 24-25]

Let n be the number of prevertices and let z_n denote the n -th prevertex. Then fix prevertices z_{n-2} , z_{n-1} and z_n to lie in order on the border of the domain. Calculate rest of the prevertices using following formulas

$$(4.8) \quad \left\{ \begin{array}{l} \frac{\left| \int_{z_j}^{z_{j+1}} w'(x) dx \right|}{\left| \int_{z_1}^{z_2} w'(x) dx \right|} = \frac{|w_{j+1} - w_j|}{|w_2 - w_1|}, \quad j = 2, 3, \dots, n-2 \\ \frac{\left| \int_{z_{J-1}}^{z_{J+1}} w'(x) dx \right|}{\left| \int_{z_1}^{z_2} w'(x) dx \right|} = \frac{|w_{J+1} - w_{J-1}|}{|w_2 - w_1|} \quad \text{if } w_J = \infty \text{ for } J < n \end{array} \right.$$

where $w'(x)$ comes from Equation (4.2) (The Schwarz-Cristoffel formula). Also we must require that no two infinite vertices be adjacent. This requirement can be achieved by introducing one degenerate vertex with the interior angle π on the straight line between infinite neighbors.[Dri02, p. 24-25]

Above side-length formulation gives $n - 3$ equations that need to be solved, for details how this can be done for unit disk see [Dri02, p. 25-27], unfortunately Driscoll has not given specific details in his book. Solution of the equations require nonlinear equations solver, for solutions of nonlinear set of equations see [NRC, chapters 9.6-9.7]. Generally speaking solving a set of nonlinear equations is hard.

The biggest drawback of the side-length method is a source of numerical error called

crowding. Crowding is present in almost all numerical conformal mapping techniques. It should be also noted that above Davis's iteration suffers from crowding too. Crowding occurs when two prevertices come too close to each other. Because of the limited numerical floating point accuracy it might be numerically impossible to even distinguish two different prevertices. Crowding especially affects regions with long and thin areas, like a long rectangles. [Dri02, p. 20-21]

For some time thin long regions were a problem and special domain decomposition methods were developed to combat crowding [Dri02, p. 21]. What is more interesting from our point of view is that Papamichael and Stylianopoulos developed domain decomposition methods especially for module computations, for further details see [PS88], [PS92] and [PS99]. The biggest drawback of these domain decomposition methods for module computations are that they only give approximate values.

Because of the ill-effects of the crowding to the numerical Schwarz-Cristoffel mapping Driscoll and Vavasis developed Cross-ratios and Delaunay Triangulations method for solving the Schwarz-Cristoffel parameter problem. This method is currently the state of the art in numerical Schwarz-Cristoffel mapping. Driscoll and Vavasis believe that CRDT (Cross-ratios and Delaunay Triangulations)-method removes crowding completely and solves convergence problems. This method is too lengthy to be described here, for details see [DV98] and [Dri02, p. 30-39].

Although Driscoll and Vavasis have developed a very good method, it is very difficult and requires lots of hard work to implement it. Luckily Driscoll has implemented this method as well as side-length method in Matlab toolbox called "Schwarz-Christoffel Toolbox" and made it available for free. We will look at Driscoll's Schwarz-Christoffel toolbox later.

Any numerical implementation of the Schwarz-Christoffel mapping needs a numerical integration method. Although almost any numerical integration method can be used, the Schwarz-Christoffel integral is often singular at prevertices and conventional methods perform poorly. The current state of the art integration method for the Schwarz-Christoffel mapping is the so called compound Gauss-Jacobi quadrature. This integration method takes account singularities at both endpoints and singularities near integration interval. The details of this integration method are too lengthy to be described here, for additional information see [Dri02, p. 27-29] for general information for the Schwarz-Christoffel integral and see [NRC, p. 147-161] for the evaluation of the Gauss-Jacobi quadrature. Implementation of this integration method is much more complex than Simpson's method or even Gaussian-Quadrature.

4.2. The module of a quadrilateral and the Schwarz-Christoffel mapping.

Now we are ready to look at how the module of a quadrilateral can be computed by using the Schwarz-Christoffel mapping. Required steps are summarized in Figure 14.

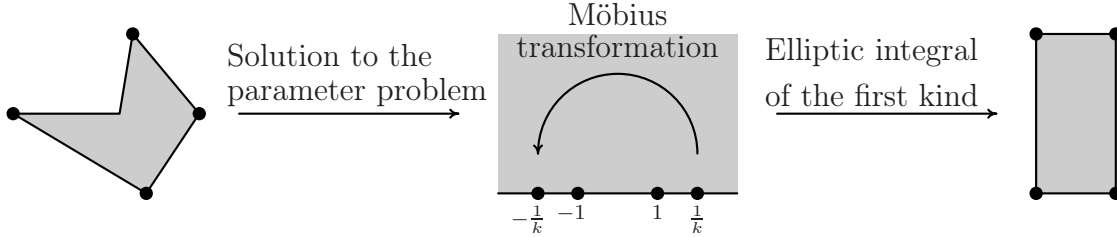


FIGURE 14. Computation of the module of a quadrilateral using the Schwarz-Christoffel mapping can be divided into smaller subtasks.

First we map the quadrilateral onto the upper half plane. This can be done by the inverse Schwarz-Christoffel mapping, however since the boundary of the quadrilateral is mapped into real axis, we need to only know the position of the vertices on the real axis. This can be accomplished by solving the SC-parameter problem. After the vertices have been mapped to upper half plane we can use elliptic integral of the first kind to map the upper half plane into rectangle. We just need to make sure that four vertices on the real axis are $\{-\frac{1}{k}, -1, 1, \frac{1}{k}\}$ for some $k \in (0, 1)$ in order to use elliptic integral. If the vertices are something else we can calculate the correct value of k by using following formula.

4.9. Theorem (Elliptic modulus formula for the upper half plane).

Let z_1, z_2, z_3 and z_4 be the positions of the vertices on the real axis, so that $z_1 < z_2 < z_3 < z_4$ and also let

$$(4.10) \quad Z = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$$

then the correct value of elliptic modulus k is given by

$$(4.11) \quad k = \frac{-2 + Z + 2\sqrt{1 - Z}}{Z}$$

Proof. The idea of the proof is to use Möbius transformation to map points z_1, z_2, z_3 and z_4 to points $-\frac{1}{k}, -1, 1$ and $\frac{1}{k}$ respectively. Then we derive the required condition for k . The Cross-ratio is invariant under Möbius transformation and we

get

$$(4.12) \quad Z = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)} = \frac{(w_4 - w_1)(w_2 - w_3)}{(w_2 - w_1)(w_4 - w_3)}$$

$$(4.13) \quad = \frac{(\frac{1}{k} + \frac{1}{k})(-1 - 1)}{(-1 + \frac{1}{k})(\frac{1}{k} - 1)}$$

$$(4.14) \quad = \frac{-4k}{(1 - k)^2} = Z.$$

We get the following equation from (4.14)

$$(4.15) \quad Zk^2 + k(4 - 2Z) + Z = 0.$$

By using the quadratic formula we get

$$(4.16) \quad k = \frac{-2 + Z \pm 2\sqrt{1 - Z}}{Z}$$

Note that Z is always negative when $z_1 < z_2 < z_3 < z_4$, so the solution is real valued. Also note that the roots are reciprocal values of each other¹². Since we need to restrict the values of k such that $\frac{1}{k} > 1$. We need to select the smallest value of k and this happens with $+$ sign in equation (4.16). \square

When the elliptic modulus k is known we can calculate the module of a quadrilateral by using complete elliptic integrals of the first kind.

4.17. Theorem (Module of a quadrilateral and complete elliptic integrals). [Dri02, p. 19]

Let Q be a quadrilateral and let k be the elliptic modulus of Q then

$$(4.18) \quad \mathbf{M}(Q) = \frac{\mathcal{K}'(k)}{2\mathcal{K}(k)},$$

where \mathcal{K} is the complete elliptic integral of the first kind and $\mathcal{K}' = \mathcal{K}(\sqrt{1 - k^2})$. See Figure 7 for the derivation of this result.

Now if the Q is the upper half plane, then we can easily calculate the value of k by using Theorem 4.9. Since it is relatively simple to calculate the module of a quadrilateral after we have mapped the quadrilateral into the upper half plane, therefore the most challenging problem is the solution to the parameter problem. See Appendix for numerical evaluation of the elliptic integrals of the first kind.

¹²Proof: $\left(\frac{-2+Z+2\sqrt{1-Z}}{Z}\right)\left(\frac{-2+Z-2\sqrt{1-Z}}{Z}\right) = \frac{(-2+Z)^2 - 4(1-Z)}{Z^2} = \frac{4-4Z+Z^2-4+4Z}{Z^2} = 1$

4.3. The Schwarz-Christoffel toolbox.

The Schwarz-Christoffel Toolbox is a Schwarz-Christoffel mapping software for Matlab by Tobin A. Driscoll. This toolbox can be downloaded from Driscoll's home page at

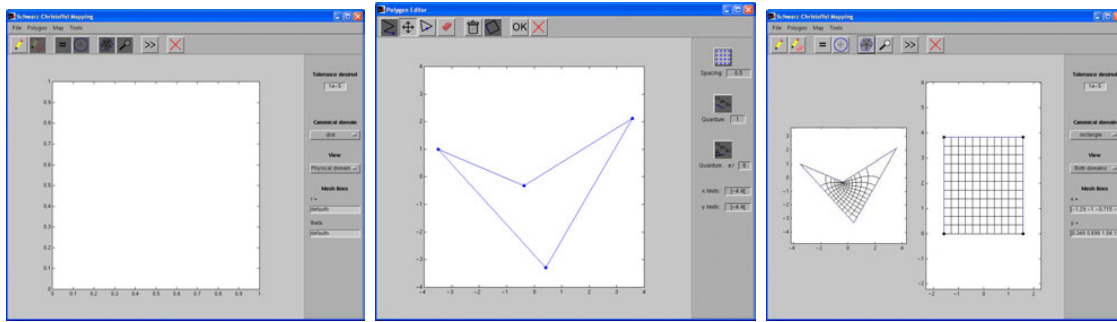
<http://www.math.udel.edu/~driscoll/SC/>.

We use the toolbox to calculate the module of quadrilaterals and next we present a brief tutorial on how to use the toolbox for this purpose.

Installation of the toolbox is straight forward, just unpack the files into directory of your choice and then add that directory to Matlab's path. This can be accomplished by the following Matlab command, provided that the toolbox is installed in `/sc` directory.

```
path(path, '/sc')
```

After the path is set the toolbox is ready for use. Toolbox contains both graphical and command line user interface. Graphical user interface(GUI) can be launched by typing `scgui`



(a) Main view

(b) Polygon editor

(c) Main view with the map

FIGURE 15. Screenshots from the graphical user interface of Schwarz-Christoffel Toolbox

Almost all functionality of the Schwarz-Christoffel Toolbox can be accessed through the graphical user interface, including creation of polygons(Figure 15(b)), selecting the domain to which the polygon is mapped and exporting data to Matlab's command line. It is important however to notice, that the toolbox GUI does not give module of the quadrilateral directly, although it is clearly visible from the final map screen (see Figure 15(c)).[SCTUG]

In this short tutorial we only concentrate on command line interface because it is more flexible and slightly harder to use. First we need to define the polygon, which we want to map. This can be done with polygon command. The following code will create polygon with vertices at 0, 4, $3 + 2i$ and $5 + 5i$. This polygon is then stored in variable p for later use.

```
p=polygon([0 4 3+2i 5+5i])
```

If we want more or different vertices we just replace or add vertices in complex representation in the list. The toolbox extends some of the most common Matlab functions(plot,eval,inv, etc.) to allow easy co-operation with the toolbox's features. So we could draw a picture of our polygon with plot(p) command. Next we need to define what kind of map we want to use. In the Schwarz-Christoffel Toolbox one selects the type of the map with the function call. We are mainly interested in rectangle maps because we know how to compute the module of a rectangle. The toolbox contains two different types of rectangle maps namely rectmap and crrectmap.

```
rectmap(p)
```

```
crrectmap(p)
```

Both above commands will map our polygon p into a rectangle with a few exceptions. First rectmap(p) command pops out a dialog where the user is requested to point out the corners of the rectangle. See Figure 16(a). Once the corners are selected then the rectmap clearly states the value of the conformal module in output feed. rectmap(p) uses side-length parameter problem solver and is therefore sensitive to crowding. It is recommended that the first two selected vertices are endpoints of a long side of the target rectangle. This improves accuracy and convergence.

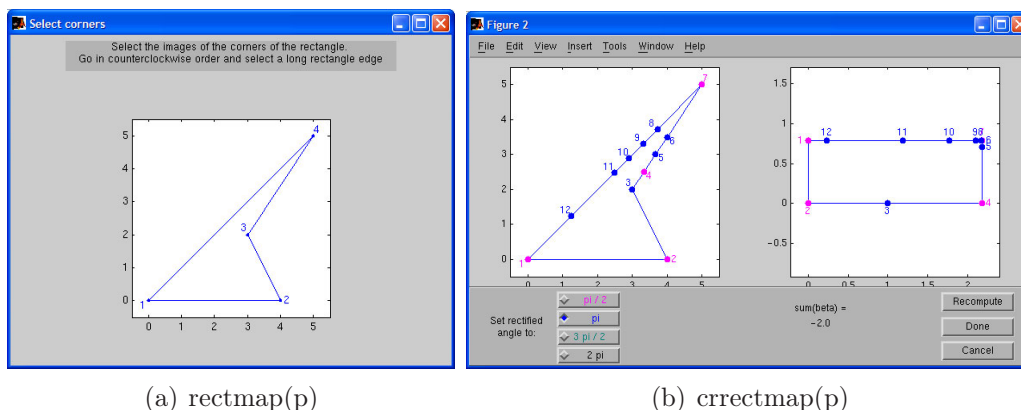


FIGURE 16. Screenshots from the rectmap and crrectmap solution process

The other command crrectmap(p) works little differently. It pops up different dialog, which can be seen from Figure 16(b). Initial polygon p is on the left side and the target rectangle is on the right side of Figure 16(b). The given vertices of p are the corners of the rectangle by default, however more advanced crrectmap(p) uses CRDT parameter problem solver and it inserts dummy vertices. These dummy vertices are blue dots while the corners of the polygon p are pink dots. CRDT algorithm uses these dummy vertices to combat crowding. Also user can change properties of any vertex(dummy or other) runtime by simply clicking on the vertex and selecting new rectifiable angle to that vertex. One should be careful with the

order of the vertices when changing vertices types, since the order of the vertices in the target polygon is determined by the vertex number and nothing else. Vertex numbering affects the Dirichlet-Neumann boundary conditions and different boundary conditions might give different results. When the changes are ready new map is computed by pressing recompute button. Done button finishes the map and return crrectmap object to command line. The biggest problem with crrectmap is that module is not clearly stated even if it is clearly visible from the picture.

Usually graphical observations are not enough and we want the most "exact" value of module to some variable for further study. The method how to do this varies with mapping function. Rectmap is simpler than crrectmap, since SC-toolbox computes its module directly. We just need to call the special modulus function to get the numerical value.

```
modulus(rectmap(p))
```

Crrectmap is more complicated since above modulus() function is not compatible with it. However we can create following Matlab function that computes the module of crrectmap objects.

Matlab function 1 Computation of the module of a quadrilateral for crrectmap objects

```
function mo = moduluscr(map)
%Computes the module of crrectmap objects
%map = mapping crrectmap object
pl2=get(map,'rectpolygon');
angpl=pl2.angle();
angpl=angpl-1;
Ipl=find(angpl);
v=[pl2.vertex(Ipl(1)) pl2.vertex(Ipl(2)) pl2.vertex(Ipl(3)) pl2.vertex(Ipl(4))];
mo=(max(imag(v)) - min(imag(v)))/(max(real(v)) - min(real(v)));
```

Once the moduluscr()-function has been created we can use it almost identically to modulus()-function, for example

```
moduluscr(crrectmap(p))
```

However popping dialog windows are impractical in the long run, since they require human interaction. We can get rid of the dialogs by giving extra parameters to rectmap()- and crrectmap()-functions. To accomplish this rectmap wants a list of vertex numbers, which are the corners of the target rectangle, as an extra input. On the other hand crrectmap takes a list of angles of the target rectangle at given vertex position. For example for polygon p

```
rectmap(p,[2 3 4 1])
crrectmap(p,[0.5 0.5 0.5 0.5])
```

Note that numbering of rectmap vertices is not [1 2 3 4]. This is because of our definition of module of a quadrilateral. We have chosen `modulus(rectmap(p,[2 3 4 1]))` to be the module of p while `modulus(rectmap(p,[1 2 3 4]))` is the reciprocal value of the module of p .

Sometimes it is necessary to know the numerical accuracy of the map. This can be accomplished with `accuracy-function` as follows.

```
accuracy(rectmap(p,[2 3 4 1]))
accuracy(crrrectmap(p,[0.5 0.5 0.5 0.5]))
```

Note that `accuracy-function` is polymorphic, so that same function applies to different types of maps.

The next Matlab function calculates the capacitance of the condenser, once the module of the quadrilateral is known. Algorithm uses the formula (3.47).

Matlab function 2 Computation of the capacitance of the quadrilateral

```
function cap=capacitance(M,tol)
%Calculates the capacitance of a condenser
%M    = The modulus of a quadrilateral
%tol  = Tolerance
n=1;
sumvalue=0;
increment=0;
if (M>0)
    while ((increment>tol) || (n<50) )
        increment=1/(n*sinh(n*pi*M));
        sumvalue=sumvalue+increment;
        n=n+2;
    end
end
cap=(1/(pi^2))*sumvalue;
```

Unfortunately there is a small bug in the version 2.3 of Schwarz-Christoffel Toolbox, which complicates computations a little bit. Evidently graphical dialogs cause `crrrectmap` to crash in some cases, these cases are not common but in a large batch they are very likely. There are two known solutions to this problem. First solution is to write `crrrectmap` call in try catch block so that if the `crrrectmap` returns an error then let `rectmap` do the computations in the catch block, this usually works but is only recommended as last resort, since `rectmap` is not as accurate as `crrrectmap`. The second better way is to disable all graphical dialogs. The following example demonstrates this bug in one particular case and shows how to get around it.

```
s=polygon([0 2 3+3i 2i])
```

```
options = scmapopt('TraceSolution','off');
crrectmap(s,[0.5 0.5 0.5 0.5],options);
```

Note that if you try just `crrectmap(s,[0.5 0.5 0.5 0.5])`, then the SC toolbox returns an error. In a large batch above solution is recommended in any case, since popping extra dialogs is waste of CPU power.

The SC toolbox is important to us because it is the main source of numerical data in this work. We present numerical data computed with Schwarz-Christoffel Toolbox later.

5. SYMMETRY PROPERTIES OF THE MODULE

Earlier we have seen how a module of a rectangle reduces to aspect ratio of the quadrilateral. Therefore we could say that the module of the quadrilateral is generalized version of the aspect ratio of the quadrilateral. In a way the aspect ratio of a rectangle is the measure of symmetries of a rectangle. In this section we investigate how we can use symmetries in the computation of the module of the quadrilateral. It turns out that there are some cases where the module computation is relatively simple thanks to symmetries.

Before we can use symmetries to calculate modules we need to present and prove Schwarz's reflection principle but first we take a look at analytic continuation. It turns out that the analytic continuation is a rather good way to extend some know module results to more general domains.

5.1. Definition (Analytic continuation). [Spi99, p. 265]

Let $F_1(z)$ be a complex function, which is analytic in a region R_1 and suppose that we can find a function $F_2(z)$ which is analytic in region R_2 such that $F_1(z) = F_2(z)$ in the region common to R_1 and R_2 . Then $F_2(z)$ is called an analytic continuation of $F_1(z)$. Also note that it suffices for R_1 and R_2 to have only a small arc in common.

In order to prove Schwarz's reflection principle we need a couple of lemmas.

5.2. Lemma. [Spi99, p. 277]

Let $F(z)$ be analytic in a region R and suppose that $F(z) = 0$ at all points on an arc PQ inside R , then $F(z) = 0$ through R .

Proof. Choose any point, say z_0 , on arc PQ . Then in some circle of convergence C with center at z_0 (This circle extending at least to the boundary of R where a singularity may exist), $F(z)$ has the Taylor series expansion.

$$(5.3) \quad F(z) = F(z_0) + F'(z_0)(z - z_0) + \frac{1}{2}F''(z_0)(z - z_0)^2 + \dots$$

But by the hypothesis $F(z_0) = F'(z_0) = F''(z_0) = \dots = 0$. Hence $F(z) = 0$ inside C .

By choosing another arc inside C , we can continue the process. In this manner we can show that $F(z) = 0$ throughout R . [Spi99, p. 277] \square

5.4. Lemma. [Spi99, p. 277]

Let $F_1(z)$ and $F_2(z)$ be analytic in a region R and suppose that on an arc PQ in R , $F_1(z) = F_2(z)$, then $F_1(z) = F_2(z)$ in R .

Proof. This follows from Lemma 5.2, by choosing $F_1(z) = F_1(z) - F_2(z)$. [Spi99, p. 277] \square

5.5. Theorem (Schwarz's reflection principle). [Spi99, p. 266]

Suppose that $F_1(z)$ is analytic in the complex region R_1 and that $F_1(z)$ assumes real values on the part of the real axis. Let us call this part LMN (see Figure 17). Then Schwarz's reflection principle states that the analytic continuation of $F_1(z)$ into region R_2 (Considered as a mirror image or reflection of R_1 with LMN as the mirror) is given by

$$(5.6) \quad F_2(z) = \overline{F_1(\bar{z})}$$

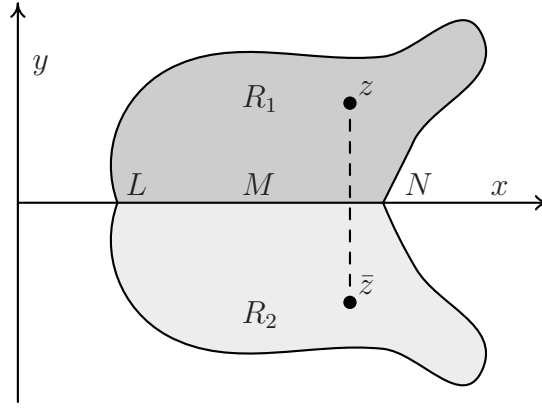


FIGURE 17. Schwarz's reflection principle

Proof. By the assumption we have on the real axis $F_1(z) = F_1(x) = F_1(\bar{x}) = \overline{F_1(\bar{z})}$. Now by Lemma 5.4 we have only to prove that $\overline{F_1(\bar{z})} = -F_2(z)$ is analytic in R_2 .

Let $F_1(z) = U_1(x, y) + iV_1(x, y)$. Since this is analytic in R_1 , we have by the Cauchy-Riemann equations.

$$(5.7) \quad \frac{\partial U_1}{\partial x} = \frac{\partial V_1}{\partial y}, \quad \frac{\partial V_1}{\partial x} = -\frac{\partial U_1}{\partial y}$$

where these partial derivatives are continuous.

Now $F_1(\bar{z}) = F_1(x - iy) = U_1(x, -y) + iV_1(x, -y)$, and so $\overline{F_1(\bar{z})} = U_1(x, -y) - iV_1(x, -y)$. If this is to be analytic in R_2 we must have, for $y > 0$.

$$(5.8) \quad \frac{\partial U_1}{\partial x} = \frac{\partial(-V_1)}{\partial(-y)}, \quad \frac{\partial(-V_1)}{\partial x} = -\frac{\partial U_1}{\partial(-y)}$$

But these are equivalent to 5.7, since

$$(5.9) \quad \frac{\partial(-V_1)}{\partial(-y)} = \frac{\partial V_1}{\partial y}$$

$$(5.10) \quad \frac{\partial(-V_1)}{\partial x} = -\frac{\partial V_1}{\partial x}$$

$$(5.11) \quad \frac{\partial U_1}{\partial(-y)} = -\frac{\partial U_1}{\partial y}$$

Hence the required result follows. [Spi99, p. 279] \square

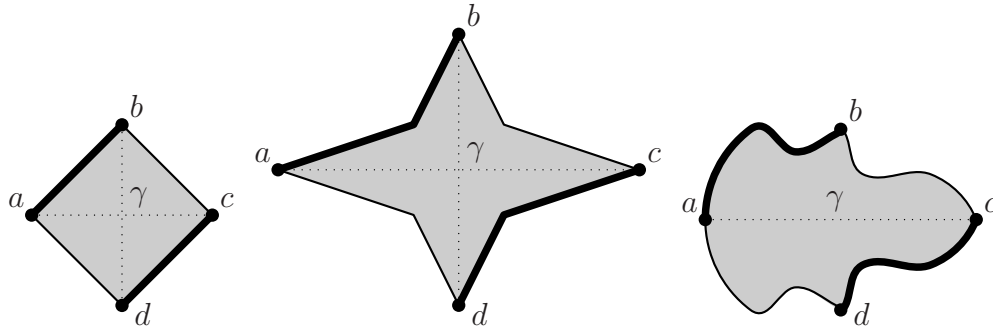
Note that the Schwarz's reflection principle can be generalized to the case where line LMN is a curve instead of a straight line segment. This generalization might produce more general results later on. For now we always assume that the reflection boundary is a straight line.[Spi99, p. 266]

5.1. Symmetric quadrilaterals.

Symmetric quadrilaterals are just normal symmetric objects with some extra requirements to boundary conditions.

5.12. Definition (Symmetric quadrilateral). [Hen91, p. 433]

The quadrilateral $(Q; a, b, c, d)$ is called symmetric if the region Q is symmetric with respect to the straight line γ through a and c , and if the points b and d are symmetric with respect to γ . See Figure 18.



(a) Square with two symmetry axes. (b) Polygon with two symmetry axes. (c) Domain with one symmetry axis.

FIGURE 18. Symmetric quadrilaterals with boundary conditions, different boundary conditions are marked as thick or thin curves. Boundary conditions are Dirichlet or Neumann conditions.

Next one is the first symmetry result, that simplifies calculations considerably.

5.13. Theorem (Every symmetric quadrilateral has module 1). [Hen91, p. 433]

Every symmetric quadrilateral has module 1.

Proof. Let Q be a symmetric quadrilateral and γ be a line that cuts Q into two halves T and T' . Let T be the half of the boundary which contains vertex d , and consider the trilateral $T(a, c, d)$. Because all trilaterals are conformally equivalent, this trilateral is conformally equivalent to the triangular trilateral with the three corners $-1, 1, i$. By the Schwarz reflection principle, this conformal mapping may be continued to a map of Q to the square quadrilateral with four corners $-1, -i, 1, i$, which obviously has module 1. [Hen91, p. 433] \square

5.14. *Remark* (Alternative definition for a polygonal quadrilaterals).

Note that for a polygonal quadrilaterals with four straight line sides, the above definition reduces to the requirement that the quadrilateral has two adjacent sides of equal length and the intersection angle of diagonals is $\frac{\pi}{2}$.

The above proof construction by Henrici is important. Later similar construction yields more symmetry results. Note that not all quadrilaterals with module one are symmetric in view of Definition 5.12. See Figure 19.

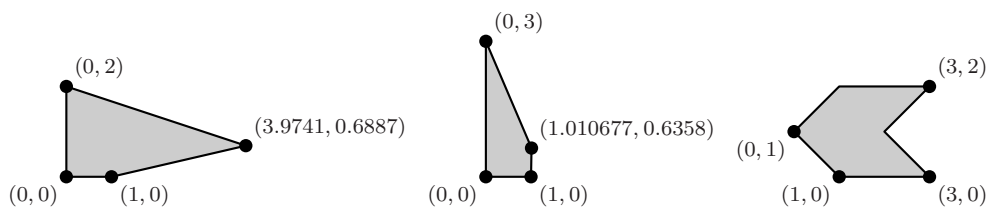


FIGURE 19. Non-symmetric quadrilaterals with module one. Note that this picture is based on numerical result and therefore contains some inaccuracy. Also note that although the last polygon is symmetric but it is not a symmetric quadrilateral

Above proof gives us a nice way to create symmetric quadrilaterals with module one.

5.15. **Theorem** (Method for creating quadrilaterals with module one).

Let $(T; a, c, d)$ be any trilateral with one straight line side. Now reflect the trilateral with straight line side acting as mirror. Then combine these two trilaterals and we get a quadrilateral with module one.

Note that this construction can be generalized for domains with arbitrary number of vertices. The only requirement is that all except one vertex are on the reflection line. See Figure 20.

Later we will see how this idea of reflection generalizes.

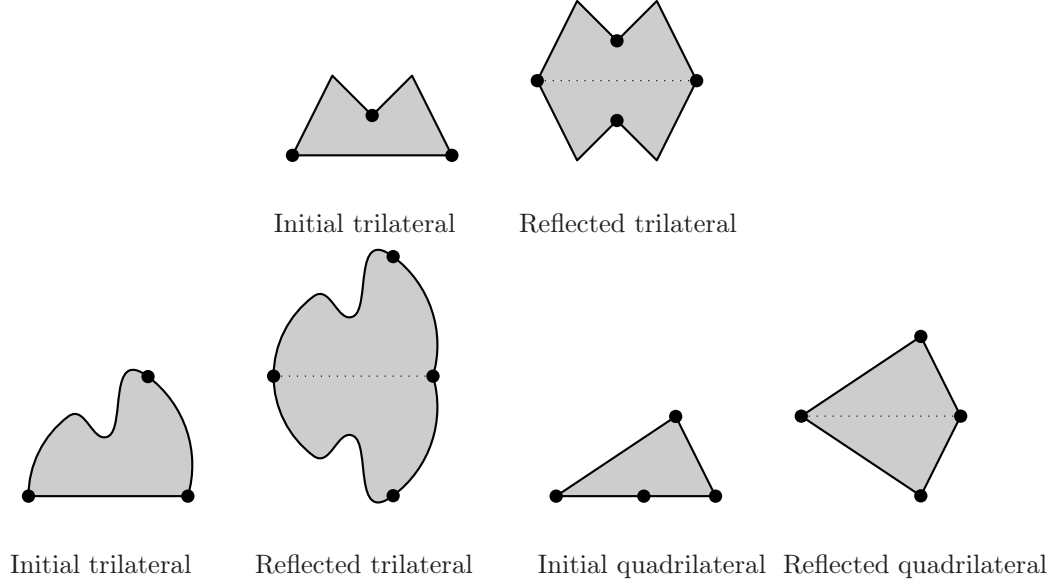


FIGURE 20. Creation of quadrilaterals with module one.

5.16. **Example** (Construction of a square and a rhombus).

It is possible to construct a square and a rhombus so that construction process keeps the module unchanged and the construction converges to a square or to rhombus. This construction gives us some insight how symmetrical quadrilaterals are formed.

Now consider the square in Figure 21(a). The square has two symmetry axis and the intersection angle of the symmetry axis is $\frac{\pi}{2}$. Now if we add another square we get a rectangle 21(b), where the symmetry axis has changed the orientation. This change of orientation is due to the fact that we did not add this extra "mass" (the new square) on the old symmetry axis. This new increment caused the old symmetry axis to change. To compensate this change in 21(b) we must move the lower right vertex so that the vertex positions are symmetrical with respect to new symmetry axis.

All other pictures are easy in the sense that symmetry axis keeps their orientation and only the number of symmetry axis changes. Note how easy it is to add mass on the symmetry axis. We just move one vertex along the symmetry axis and we are done. We could even add smaller squares or some other symmetrical forms on the symmetry axis. Also note that we can add the same amount of mass to different sides of the symmetry axis. This can be seen from pictures 21(a) and 21(c), where we have added two squares on the different sides of the same symmetry

axis symmetrically¹³. It should be emphasized that this works only for symmetric quadrilaterals, like those in Figure 21. Also note there is one picture missing from the construction given below. The picture is obviously square x4, which is just a square. We can add multiple items at the same time too. For example we could easily get from 21(d) to 21(h) by inserting four squares in the corners of 21(d).

For a rhombus this construction is similar, however apparently there does not exist rhombus x2 object, that could be used in our construction easily. This is to be expected since rhombus is more general form of the square. Note that this construction is also possible for kites and concave symmetric quadrilaterals but then we need to use at least two different kinds of building blocks.

¹³This construction might give a way to tile a square, when we use different block sizes. A interesting fact is that the connection between the square tilings and the conformal module have been studied previously by Cannon, Floyd and Parry, see [CFP96].

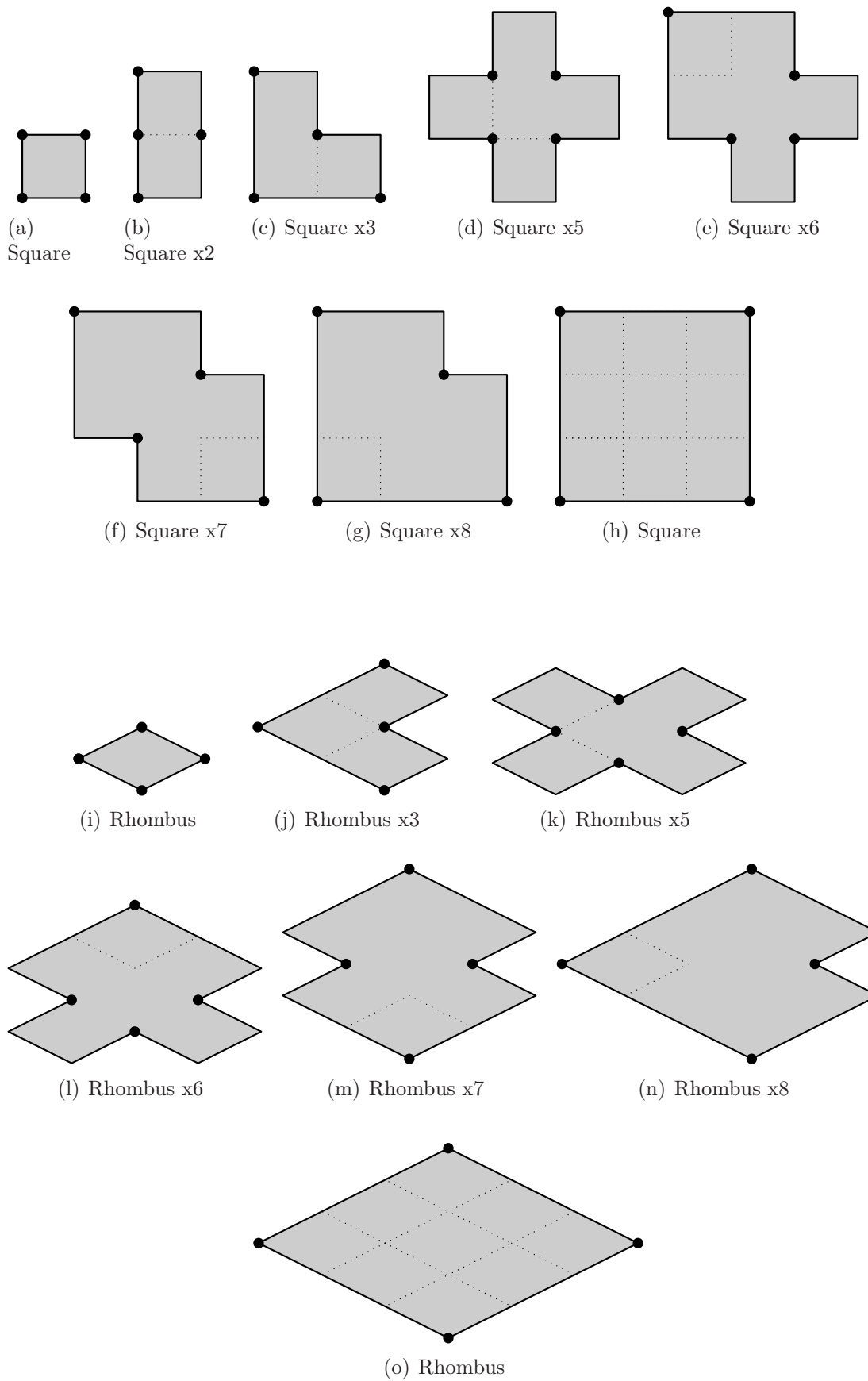


FIGURE 21. Construction of a square and a rhombus by using squares and rhombuses. Note that all quadrilaterals have module one.

5.17. Conjecture (Border rotation Theorem).

Let $Q(a, b, c, d)$ be a quadrilateral with a symmetric domain. Let $2s$ be the length of the boundary curve. Then if we move each vertex along the boundary to same direction so that the total length of the movement is s then the module remains unchanged.

5.18. Remark (About the proof of the border rotation Theorem).

Note that above Theorem holds for symmetric quadrilaterals and rectangles¹⁴.

Rectangles have two sides of equal length. Let the length of these sides be x and y respectively. Then the half length of the border curve is $x + y$. Now if we move each vertex to same direction amount of $x + y$ length units then we end up just rotating vertices in the corners.

Let $Q(a, b, c, d)$ be a symmetric quadrilateral. We can rotate the quadrilateral so that the symmetry axis is the real line. Now since symmetry axis goes through two points a and c and the length of the curve above and below the real axis is the same because of the symmetry. Let us say that the half length of the border curve is $x + y$. Now the two points a and c map to each other if we move them half length of the border along the border. Now the other two vertices are above or below the real axis. Let γ_{up} and γ_{down} be part of the border curve that are above and below the real axis respectively. Now let us assume that b is in the upper half plane and d is in the lower half plane. Now the vertices b and d both divide curves γ_{up} and γ_{down} into two pieces of length x and y (so that the half length is $x + y$ and $d(a, b) = x$ and $d(b, c) = y$ along the curve). Now if we move vertex b it will end up in lower half plane and distance from the vertex a along the border curve is y . Now if we do the same thing to the vertex d it first moves a distance y to real axis and then distance x to upper half plane. Now the distance $d(a, d) = x + y - x = y$ so that both vertices b and d are at equal distance from vertex a , hence they are symmetrical respect to real axis, since the quadrilateral was symmetrical. Hence the quadrilateral is symmetric and the module remains unchanged.

5.19. Example (Creation of all symmetric quadrilaterals with four straight line sides).

We can use above construction to create all symmetric polygonal quadrilaterals with four straight line sides. Please refer to Figure 22. Note the slightly larger point in the pictures. This point starts to move from the right to the left and we keep reflecting the domain and so we get different types of symmetric quadrilaterals. Note when we reach rhombus(or square) and if we would continue our travel from the right to the left we would start to get same quadrilaterals again.

¹⁴The assumption that the above conjecture holds comes from the fact that it holds for rectangles and for some other domains. It could turn out that the above conjecture is not universally valid and requires additional assumptions. In the numerical results section we present a non-trivial domain for which it holds. This topic would require more study.

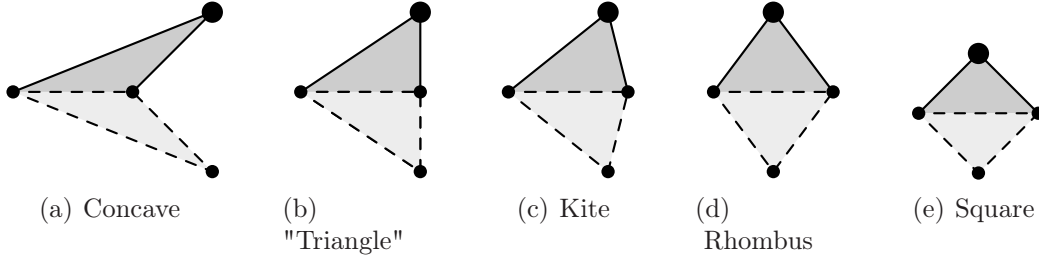


FIGURE 22. Creation of symmetric polygonal quadrilaterals with four straight line sides.

It is also good to notice that "triangle", rhombus and square are special cases of kite. While square is special case of rhombus. So the set of symmetric quadrilaterals with four straight line sides consists of symmetric concave quadrilaterals and kites.

5.20. Theorem.

Let Q be polygonal quadrilateral $(Q; a, b, c, d)$ with four straight line sides. The module of the quadrilateral Q is then determined by two vertices.

Proof. Since the module of the quadrilateral is a conformal invariant we can rotate, translate and scale one side of the quadrilateral so that it is on closed interval $[0, 1]$ on the real axis, such that quadrilateral has one vertex at $x = 0$ and other one at $x = 1$. Now we can use above technique to all polygonal quadrilaterals with four straight line sides and reduce them to $Q(0, 1, c', d')$. Since not all quadrilaterals are conformally equivalent then the module must change as vertices c' and d' changes. \square

Note that the vertices and the boundary of quadrilateral, contain some additional information that is irrelevant when calculating the module. The difficulty is in finding out what is useless information and what is not.

5.2. Reflection properties of the module of a quadrilateral.

The next theorem gives us a nice way to calculate modules of some symmetric quadrilaterals.

5.21. Theorem (Reflection symmetry).

Let Q be quadrilateral $(Q; a, b, c, d)$ with module M and let Q' be the mirror image of the quadrilateral Q through some straight boundary line with two vertices. If we combine quadrilateral Q and its mirror image Q' then the module of the new quadrilateral is $2M$ or $M/2$ depending on the reflection boundary as follows. If we reflect through the Dirichlet boundary the module is doubled and reflection through the Neumann boundary halves the module. Vertices on the reflection boundary are removed and other vertices are reflected through the reflection boundary.

Proof. We use similar construction that was used by Henrici to prove that the symmetric quadrilaterals have module one. Proof is based on graphical insight. Please

see Figure 23.

Let $Q = (Q; a, b, c, d)$ be the quadrilateral to be reflected, as above. Now by assumption quadrilateral Q has one straight line as its boundary curve. We can place this boundary line on the x -axis by a simple conformal rotation. Since quadrilateral Q has module M we can map the quadrilateral Q conformally to rectangle with side lengths 1 and M respectively. Now by the Schwarz reflection principle we can extend this map to its mirror image too. We get a map from domain $Q \cup Q'$ to rectangle with side lengths 1 and $2M$ or 2 and M . The side lengths depend on the reflection boundary as follows.

- Reflection boundary is the Neumann boundary.
If the reflection boundary is the Neumann boundary, then we must orient our rectangle in the same way because of the Schwarz reflection principle requires that on the reflection boundary both functions must take same values. For a rectangle this means that the side lengths of the rectangle are 2 and M . Now if we calculate the module of the rectangle by formula (3.49) we have $M/2$.
- Reflection boundary is the Dirichlet boundary.
If the reflection boundary is the Dirichlet boundary, then we must again orient our rectangle in same way as above. This time, however, the side lengths of the rectangle are 1 and $2M$. Now if we calculate the module of this rectangle by (3.49) we get $2M$.

Since $Q \cup Q'$ and rectangle are conformally equivalent we can conclude that the module of the quadrilateral $Q \cup Q'$ is $2M$ or $M/2$ depending on the reflection boundary. \square

Note that our form of Schwarz's reflection principle requires that the reflection boundary is a straight line. Proof might be generalized to case where reflection boundary is some kind of a curve. Also note that if we do two reflections one through the Dirichlet boundary and other one through the Neumann boundary(or other way around) then the module of the resulting quadrilateral remains unchanged.

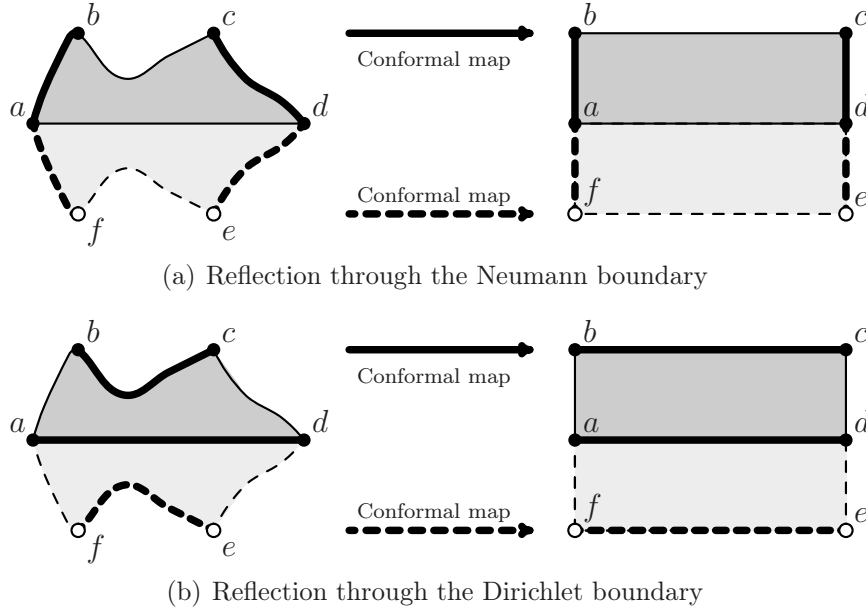


FIGURE 23. Proof construction for reflection symmetry. The Dirichlet boundary conditions are marked as thick lines. The Neumann boundary conditions are thin lines.

Note that doubling the area(reflection) can have different effect on the module of the quadrilateral depending on the boundary. Also since the module of a quadrilateral is not preserved in reflection, then it is not a conformal map.

5.22. Example (Reflections of a square).

A square is the most basic component in the module computations. As we know already that a square has the module one. Let us look how a square behaves under reflections. See Figure 25

- (a) When we reflect a square one might think that there is no difference if we reflect square through Neumann or Dirichlet boundary. We get same rectangle in both cases, however the boundaries of the resulting rectangle are different so we must be careful with the reflection boundaries, see Figure 24. Now let us reflect through the Neumann boundary, see Figure 25(a).
- (b) We reflected the square through the Neumann boundary and the module of the resulting quadrilateral is $\frac{1}{2}$ by the reflection symmetry Theorem. Now let us reflect again, now through the Dirichlet boundary, see Figure 25(b).
- (c) See Figure 25(c). We reflected the rectangle through the Dirichlet boundary and the module of the resulting quadrilateral is 1 by the reflection symmetry

Theorem. Note that two reflections through the Dirichlet and the Neumann boundaries left the module unchanged.

- (d) After the last reflection module is halved again. Now we could keep reflecting forever, however we would only get rectangles or squares and these kinds of reflections are not very interesting in the long run. See Figure 25(d).

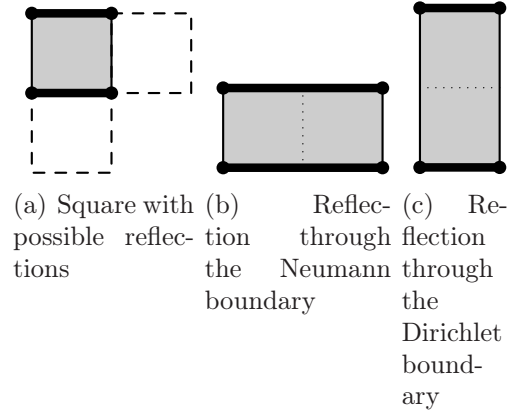


FIGURE 24. Reflections of square. The Dirichlet boundary conditions are marked as thick lines. The Neumann boundary conditions are thin lines.

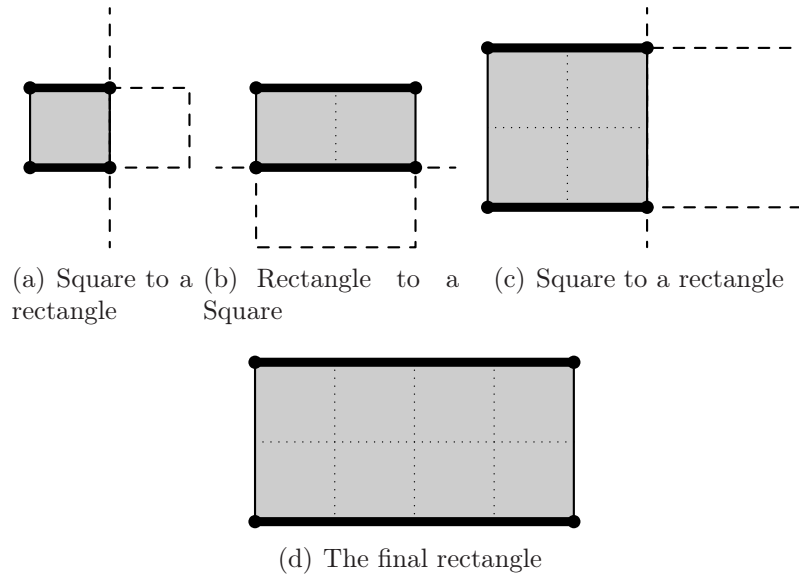


FIGURE 25. More reflections of square. The Dirichlet boundary conditions are marked as thick lines. The Neumann boundary conditions are thin lines.

5.23. Example (Reflections leading to Octagon).

This problem setup is taken from [Hen91, p. 443] exercise 5.

Let a, b, c, d be the corners of a rhombus R with center at 0 . Reflect each triangle $0ab, 0bc, 0cd, 0da$ at its hypotenuse, thereby obtaining an irregular octagon Q with corners a, e, b, f, c, g, d, h . Show that the $M(Q; e, f, g, h) = 1$ See Figure 26(a)[Hen91, p. 443].

Note that the polygon $(a, 0, d, h)$ is a kite. We can modify this kite and make it symmetric. See the modified kite in Figure 26(b). Now we reflect the modified kite through x -axis. See the resulting Figure in 26(c). Last we reflect the quadrilateral through y -axis(see Figure 26(d)). Since we have done two reflections one with the Dirichlet boundary and one with the Neumann boundary we can conclude that the module of the resulting octagon is the same as the module of our modified kite. So module of the octagon is one.

Henrici also noted that the harmonic measure of Q is $\frac{1}{8}$.

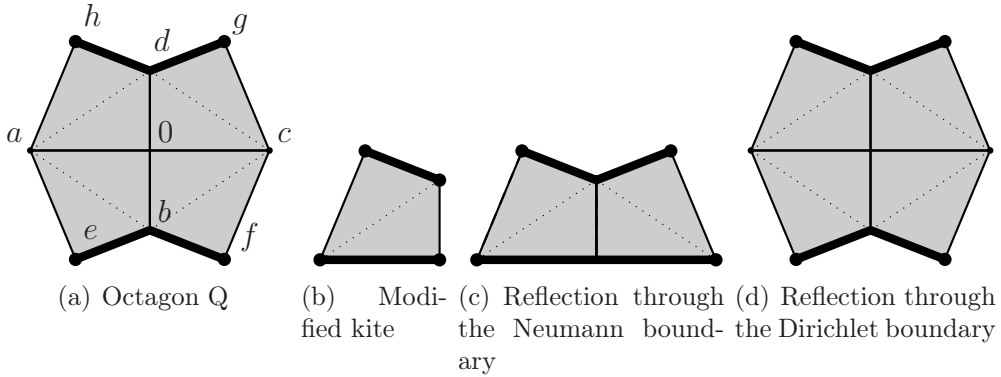


FIGURE 26. Reflections of kite. The Dirichlet boundary conditions are marked as thick lines. The Neumann boundary conditions are thin lines.

Note how reflecting increase or decrease the module value by factor of 2. So all possible module values are of the form

$$(5.24) \quad M(Q') = M2^n \quad n \in \mathbb{Z}.$$

This naturally rises the question, what are the odd numbered values. The answer is quite simple. Note how the same initial shape is repeating over and over again. For example see how rectangles and squares(dotted lines) repeat in Figure 25.

5.25. Theorem (Generalized reflection symmetry).

Let Q be a quadrilateral and if we duplicate it by reflecting and copying the original quadrilateral like in Figure 27, then the resulting quadrilateral has module $nM(Q)$

or $1/nM(Q)$, depending on the reflection boundary, where n is an integer which tells the total number of repeated quadrilaterals.

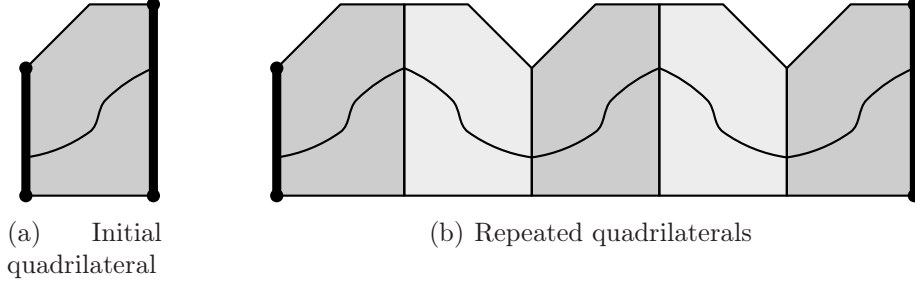


FIGURE 27. Quadrilaterals with module line curves. The Dirichlet boundary conditions are marked as thick lines. The Neumann boundary conditions are thin lines.

Proof. The proof is based on the Figure 27. Let Q_i be the initial quadrilateral with module M and extremal metric ρ^* . Since we know the extremal metric ρ^* , we can calculate the ρ^* -length L_{ρ} of the module line curve and ρ^* -area A_{ρ} for the Q_i . This quadrilateral with one module line curve is in the Figure 27(a). Since we know the extremal metric ρ^* and the exact values of A_{ρ} and L_{ρ} , then the reciprocal module formula¹⁵ (3.9) gives the exact results and we get

$$(5.26) \quad M(Q_i) = \frac{L_{\rho}^2}{A_{\rho}}.$$

Now if we start to duplicate the initial quadrilateral like in Figure 27(b), then the total area of the resulting quadrilateral is nA_{ρ} , where the n is an integer, which tells the number of the repeated initial quadrilaterals. Also it is simple to calculate the length of the module line in the resulting quadrilateral, since the module curve is repeated, too, and it still is the shortest possible curve. The total length of the module line curve of the resulting quadrilateral is nL_{ρ} . Now we plug these values into (5.26) and we get for the resulting quadrilateral

$$(5.27) \quad M(Q) = \frac{(nL_{\rho})^2}{nA_{\rho}} = \frac{nL_{\rho}^2}{A_{\rho}} = nM(Q_i).$$

□

5.3. Ring domains and the module of a quadrilateral.

The Riemann's mapping theorem is intended for simply connected domains only but something can be said about conformal equivalence of multiply connected domains. It is possible to extend Riemann's mapping theorem for domains with one hole. In

¹⁵We use the reciprocal version of the module formula because of the picture 27.

this case the multiply connected domain is mapped into a region bounded by two concentric circles. Consider two annuli Ω_1 and Ω_2

$$(5.28) \quad \Omega_1 = \{z \mid r_1 < |z| < R_1\}, \quad \Omega_2 = \{w \mid r_2 < |w| < R_2\}$$

Now there exist a conformal map $F : \Omega_1 \rightarrow \Omega_2$ if and only if the ratios R_1/r_1 and R_2/r_2 are equal [Spi99, p. 202] [Fla83, pp. 325-326]. The above extension of the Riemann's mapping theorem allows us to extend the module concept into multiply connected domains with one hole. These domains are called ring domains. First we need to investigate how the module of a quadrilateral is related to the module of a ring domain. This is shown in Figure 28.

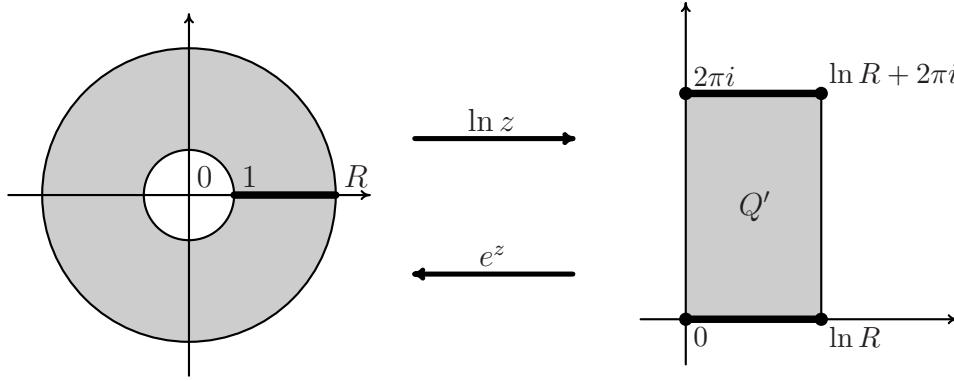


FIGURE 28. A conformal mapping from a rectangle to a ring domain and vice versa. The figure shows the relation between the module of quadrilateral and the module of a ring domain.

By using Figure 28 we can now define the module of a ring domain.

5.29. Definition (Module of the ring domain). [Küh05, p. 101]

Let $\Omega = \{z \mid r < |z| < R\}$ be an annulus, then the module of the ring domain Ω is

$$(5.30) \quad M(\Omega) = \frac{2\pi}{\ln\left(\frac{R}{r}\right)}$$

Sometimes working with ring domains have its advantages, for example capacitance formula for a annulus is rather simple.

5.31. Theorem (Capacitance formula for an annulus). [Spi99, pp. 255-256]

Let $\Omega = \{z \mid r < |z| < R\}$ be an annulus, then the capacitance of the ring domain Ω is

$$(5.32) \quad C = \frac{1}{2 \ln\left(\frac{R}{r}\right)} = \frac{M(\Omega)}{4\pi}$$

Above equation is valid if there is a vacuum between the conductors. If there is a medium of dielectric constant κ between the conductors, we must divide the above equation by κ .

Proof. For the proof see [Spi99, pp. 255-256]. □

We can use reflection properties to get ring domains too. See Figure 29.

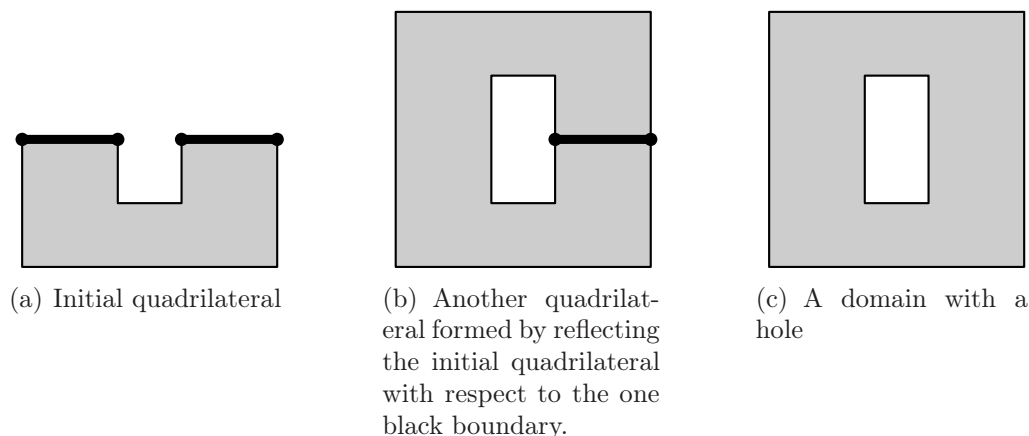


FIGURE 29. Reflections leading to domains with holes. Note that in the reflection process one loses the other boundary condition boundary.

If we attempt to extend the Riemann's mapping theorem even further, then we are in a trouble. It turns out that in three dimensional space the family of conformal mappings is very restricted and contains, essentially, only Möbius transformations. This difficulty has lead to development of quasiconformal mappings. Quasiconformal mappings are not conformal but they differ from a conformal map only some fixed amount. The hope is that some of the conformal mapping theory can be taken to higher dimensions with quasiconformal mappings. Quasiconformal mappings can be defined with the help of the module of a quadrilateral.

6. COMPUTATIONAL RESULTS

In this section we present some computational observations and some figures and try to interpret them according to presented theory. This section is a numerical section and we do not prove anything and all results should be viewed with caution.

All the results in this chapter are computed with Matlab's Linux version 7.1.0.183 (R14) with service pack 3 and the Schwarz-Christoffel toolbox version 2.3¹⁶.

6.1. Additive quadrilaterals.

The above generalized reflection symmetry theorem, explains some additive properties of the module of a quadrilateral.

¹⁶The data for the Figure 19 has been created with this same setup.

6.1. Definition (Additive quadrilaterals).

Let Q_1 and Q_2 be quadrilaterals with conformal modules M_1 and M_2 respectively. Now if we join the quadrilaterals Q_1 and Q_2 according to some rule R so that resulting quadrilateral is connected and has conformal module $M_1 + M_2$, then we say that quadrilaterals Q_1 and Q_2 are additive with the rule R .

Generally speaking the module of a quadrilateral is not additive, however there are many exceptions to this. Above we have seen how the reflection can be seen as an additive property of quadrilaterals. However not necessarily all additive quadrilaterals are created like that.

6.2. Example (Rotation and glue for identical quadrilaterals).

This example is partially based on a numerical observation. It was natural to test whether the reflection Theorem could be extended to include rotated and glued domains (see Figure 30). This turned out to be indeed unsuccessful, however the amount of the error seemed to be proportional to the length of the glued side. Now the proof of the generalized symmetry Theorem gives us a one possible explanation to this behaviour. Let us assume that the initial quadrilateral with one module line is given in Figure 30(a). Now notice if we just rotate and glue the domains together then the two module line curves do not touch (Figure 30(b)) and hence the reflection theorem cannot give accurate results. Apparently however if we make the length of the glued side smaller and smaller then the distance between the two curves approach zero and the error term might approach zero as well (this was indeed observed for some domains). As the glued side gets shorter and shorter then the extremal metric for the combined domain approaches the extremal metric of the initial domain.

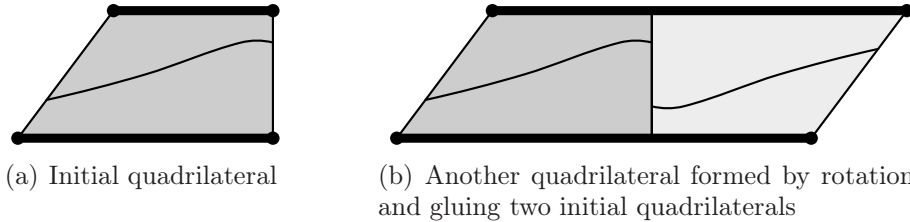


FIGURE 30. Non-additive quadrilaterals formed by rotation and gluing. The Dirichlet boundary conditions are marked as thick lines. The Neumann boundary conditions are thin lines.

Sometimes the reason for the additivity is well hidden. Consider the following example.

6.3. Example (Additive quadrilateral according to numerical data).

Let D be a polygonal domain with vertices at $i, 1, 3, 2 + i, 3 + 2i$ and $1 + 2i$ and let $Q(D, i, 1, 3, 3 + 2i)$ to be a quadrilateral. Now form a new quadrilateral Q' by repeating the original quadrilateral Q by N -times. Then we get a polygonal domain D' with vertices at $i, 1, 2N + 1, 2N + i, 2N + 1 + 2i$ and $1 + 2i$. Now numerical values suggest that

$$(6.4) \quad M(Q'(D', i, 1, 2N + 1, 2N + 1 + 2i)) = N, \quad \forall N \in \mathbb{N}.$$

This process is presented in Figure 31 and numerical data is presented in Table 1.

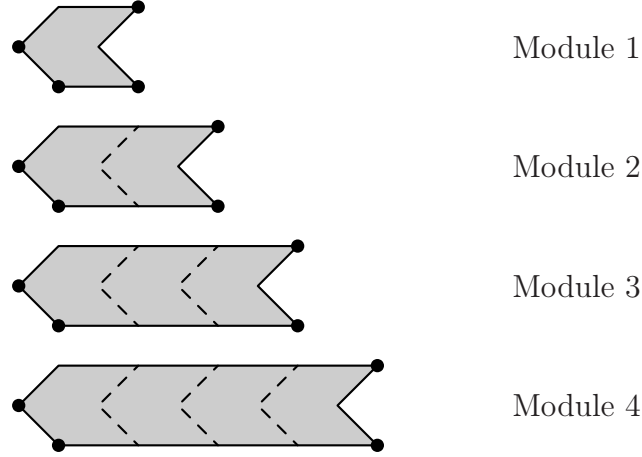


FIGURE 31. Numerical values suggest that the above quadrilateral is non-symmetric additive quadrilateral that is not formed by reflection properties.

N	Module	Apparent accuracy
1	0.99999999948263	1.18e-08
2	2.00000000266110	2.31e-09
3	3.00000003040984	4.69e-08
4	3.99999993341716	3.34e-08
8	8.00000001352554	3.21e-08
16	15.99999923586680	1.69e-08
32	31.99999802973663	1.12e-07
64	63.99999918946635	6.76e-07
128	127.9999937676023	5.38e-08

TABLE 1. Numerical data for the Figure 31 and some additional values. Variable N is the number of repeated quadrilaterals.

The following Matlab script produces the data in Table 2 with different values of N . The script uses `moduluser()`-function from the SC toolbox section. One can produce these values with the help of the graphical interface too.

```

format long;
N=1;
p=polygon([i 1 2*N+1 2*N+i 2*N+1+2i 1+2i]);
map=crrectmap(p,[0.5 0.5 0.5 1 0.5 1]);
moduluscr(map)
accuracy(map)

```

The above additive property naturally rises the question about the existence of another reflection symmetry formula(or non-straight line mirror), which could explain the above behaviour¹⁷.

Furthermore the above quadrilateral has other interesting properties that are summarized in Figure 32.

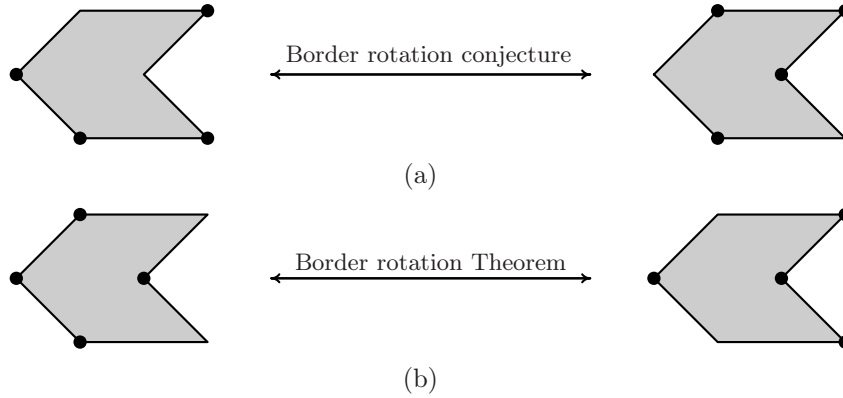


FIGURE 32. Numerical values suggest that above equalities hold and all quadrilaterals have module one. Note that last two quadrilaterals have clearly module one, since they are symmetric and rotation Theorem holds for symmetric quadrilaterals.

It seems that border rotation conjecture might hold for above quadrilateral and since the domain is symmetric we can easily make the quadrilateral symmetric by moving its vertices. Now the question is that is there a way to move the vertices on the border so that we could get from Figure 32(a) to Figure 32(b) and prove it.¹⁸

It is interesting to note that if we apply border rotation conjecture to Figure 31 we get another additive quadrilateral.

¹⁷This property might be explained simply by border rotation conjecture and the fact that the area grows fixed amount in every step. Border rotation conjecture would allow the border value correspondence that is needed in order to use Schwarz reflection principle. See figure 32

¹⁸It is a relative simple task to find a model in which vertices move like above.

6.5. **Example** (Additive quadrilateral according to border rotation conjecture).

Let D be a polygonal domain with vertices at $i, 1, 3, 2+i, 3+2i$ and $1+2i$ and let $Q(D, 1+2i, 1, 2+i, 3+2i)$ to be a quadrilateral. Now form a new quadrilateral Q' by repeating the original quadrilateral Q by N -times. Then we get a polygonal domain D' with vertices at $i, 1, 2N+1, 2N+i, 2N+1+2i$ and $1+2i$. Now numerical values suggest that

$$(6.6) \quad M(Q'(D', 1+2i, 1, 2+i, 3+2i)) = N, \quad \forall N \in \mathbb{N}.$$

This process is presented in Figure 33 and numerical data is presented in Table 2.

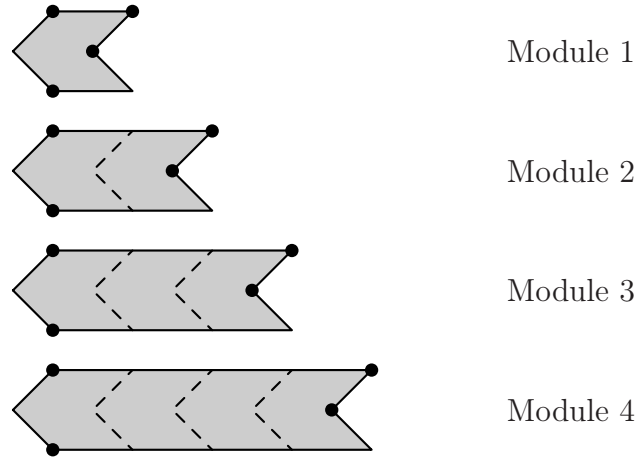


FIGURE 33. Numerical values and border rotation conjecture suggest that the above quadrilateral is non-symmetric additive quadrilateral.

N	Module	Apparent accuracy
1	1.00000000154021	1.07e-08
2	1.9999999973254	2.31e-09
3	2.99999994098051	4.41e-08
4	3.99999987501554	3.34e-08
8	8.00000001398040	3.20e-08
16	15.99999954897752	3.11e-08
32	31.99999865672795	1.57e-07
64	63.99999888846909	6.76e-07
128	127.9999921111690	5.38e-08

TABLE 2. Numerical data for the Figure 33 and some additional values. Variable N is the number of repeated quadrilaterals.

The following Matlab script produces the data in Table 2 with different values of N . The script is very similar to the above script, except that the vertices are rearranged.

```

format long;
N=1;
p=polygon([1+2i i 1 2*N+1 2*N+i 2*N+1+2i]);
map=crrectmap(p,[0.5 1 0.5 1 0.5 0.5]);
moduluscr(map)
accuracy(map)

```

6.2. The module of a quadrilateral under movement of one vertex.

Next we investigate what happens to the module of a quadrilateral if one of its vertex starts to move, while other vertices remain still¹⁹. The Matlab source code for the creation of the numerical data and plotting script for the data is available in the appendix. All the data was created by the following function call with parameters given below, where $n \in \{0.5 \ 1 \ 2 \ 3 \ 4 \ 5\}$ ²⁰.

```
imagematrix=computeData(151,8,n);
```

Consider the situation in Figure 34. The original quadrilateral is displayed as a blue rectangle. Vertices on the axis remain still, while the remaining vertex moves to some point P . The background color at point P shows the module of the quadrilateral with moving vertex at point P . Also since the background color is somewhat hard to interpret we have added some contour lines. These contour lines show paths for which module remains unchanged under the movement of one vertex. The value of the constant module is displayed on the contour line.

It is interesting to note that in Figure 34(b) module of the quadrilateral gets reciprocal values on the different sides of the symmetry axis $y = x$ (for example see lines 1.11 and 0.9, $\frac{1}{1.11} \approx 0.9$). This is to be expected, since the resulting quadrilaterals have the same domain and reciprocal boundary conditions. Next we list some general observations from Figure 34.

- Note how the module is continuous in the sense that the constant module curve for the module of the original rectangle begins at the corner of the every rectangle.
- Note that as we travel further and further away from the original rectangle we need to make bigger changes in the position of the moving vertex to get the same effect as moving near the original rectangle.
- Note that the constant module lines are curved to up or down, depending on the position of the curve and the original quadrilateral. What makes this even more interesting is the fact that apparently there always exist a curve which has a zero curvature(a line). In Figure 34(b) this curve with zero curvature is the symmetry axis. The zero curvature line in general has

¹⁹The module of a quadrilateral under the movement of one vertex have been studied before by Dubinin and Vuorinen in [DV05].

²⁰Total 23104 Schwarz-Christoffel transformations were calculated for each data set(different values of n) and hence for each picture. This process took about 2-4h on a standard shared workstation.

interesting properties. Consider Example 5.16, where it was relatively simple to increase mass on the symmetry axis and move the vertex along the axis. This rises a question that could the zero curvature line be considered as a generalized symmetry axis? This generalized symmetry axis would not need to be symmetric since it needs to grow in a unbalanced way to keep module different from 1. Also note that if the height of the rectangle is less than 1, then the zero curvature line is above the line $y = x$. On the other hand if the height of the rectangle is greater than 1 then the zero curvature line is below the line $y = x$. For example compare figures 34(a) and 34(c).

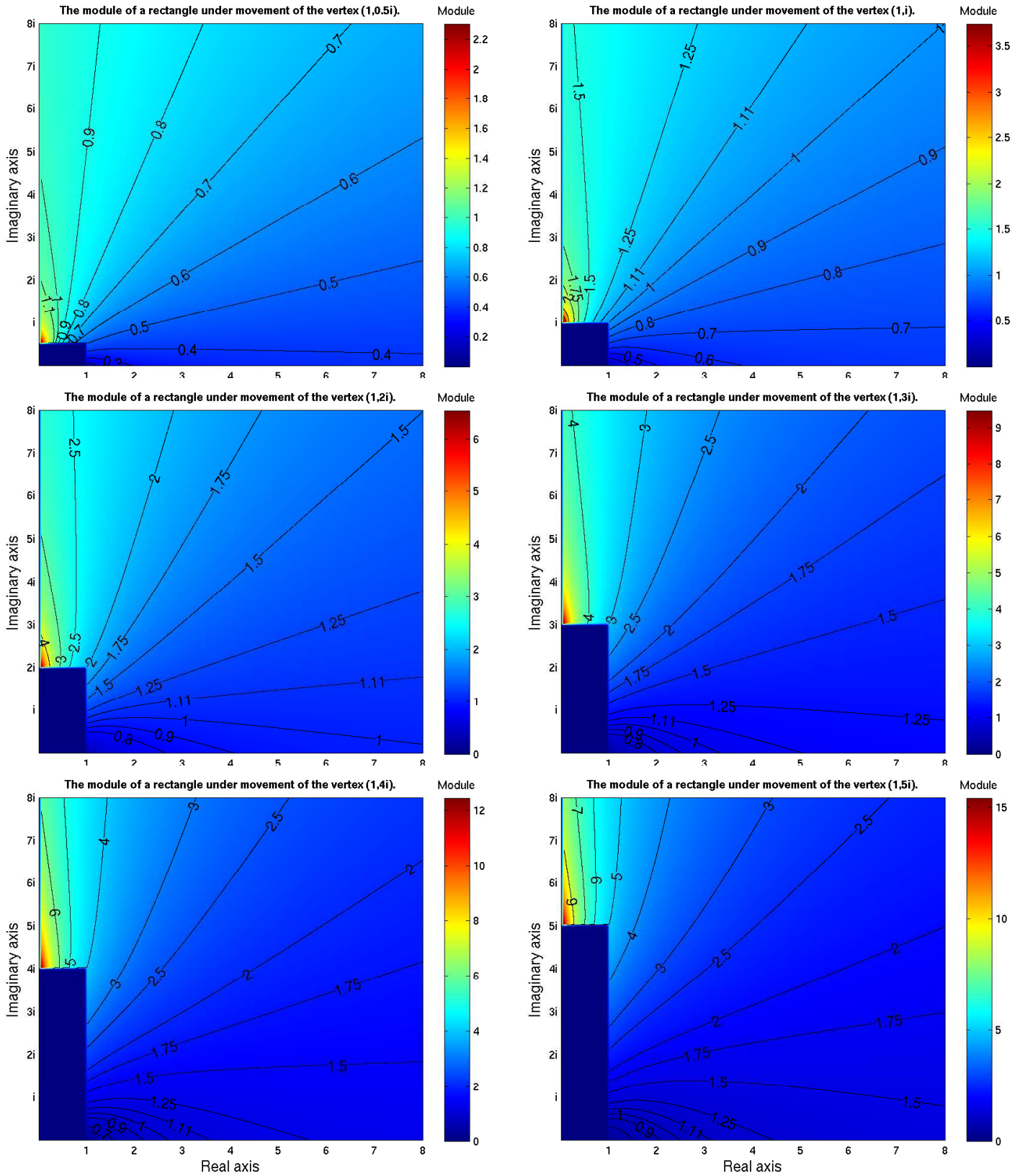


FIGURE 34. The module of a rectangle under movement of one vertex.

The next Figure 35 shows that the capacitance of a quadrilateral behaves more or less like the reciprocal value of the module of a quadrilateral.

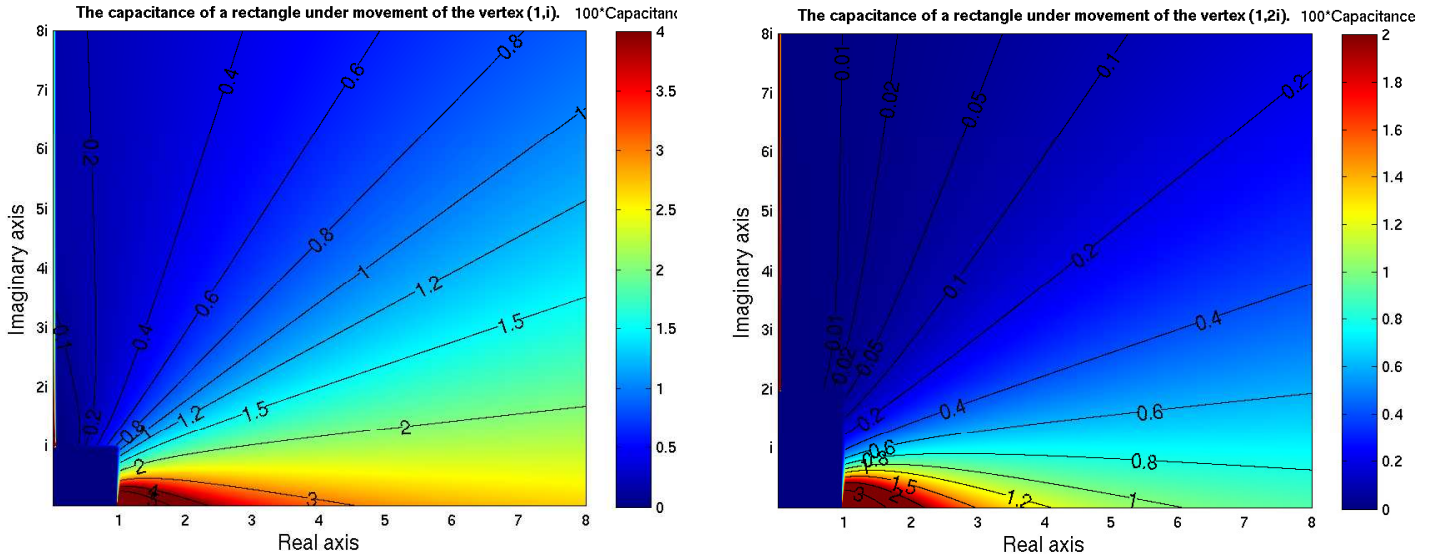


FIGURE 35. The capacitance of a rectangle under movement of one vertex. Note that the value of the capacitance is multiplied by a factor of 100.

Next we look at how the area, the perimeter and the angle between the diagonals of a quadrilateral affects the module of a quadrilateral. It is especially interesting to note the Figure 38(b), where the contour curves almost cross at one point. The scaling constant $\frac{\pi}{2}$ has been chosen so that it demonstrates this. This same behaviour can be seen in Figure 39 but the determination of a good scaling constant is more difficult.

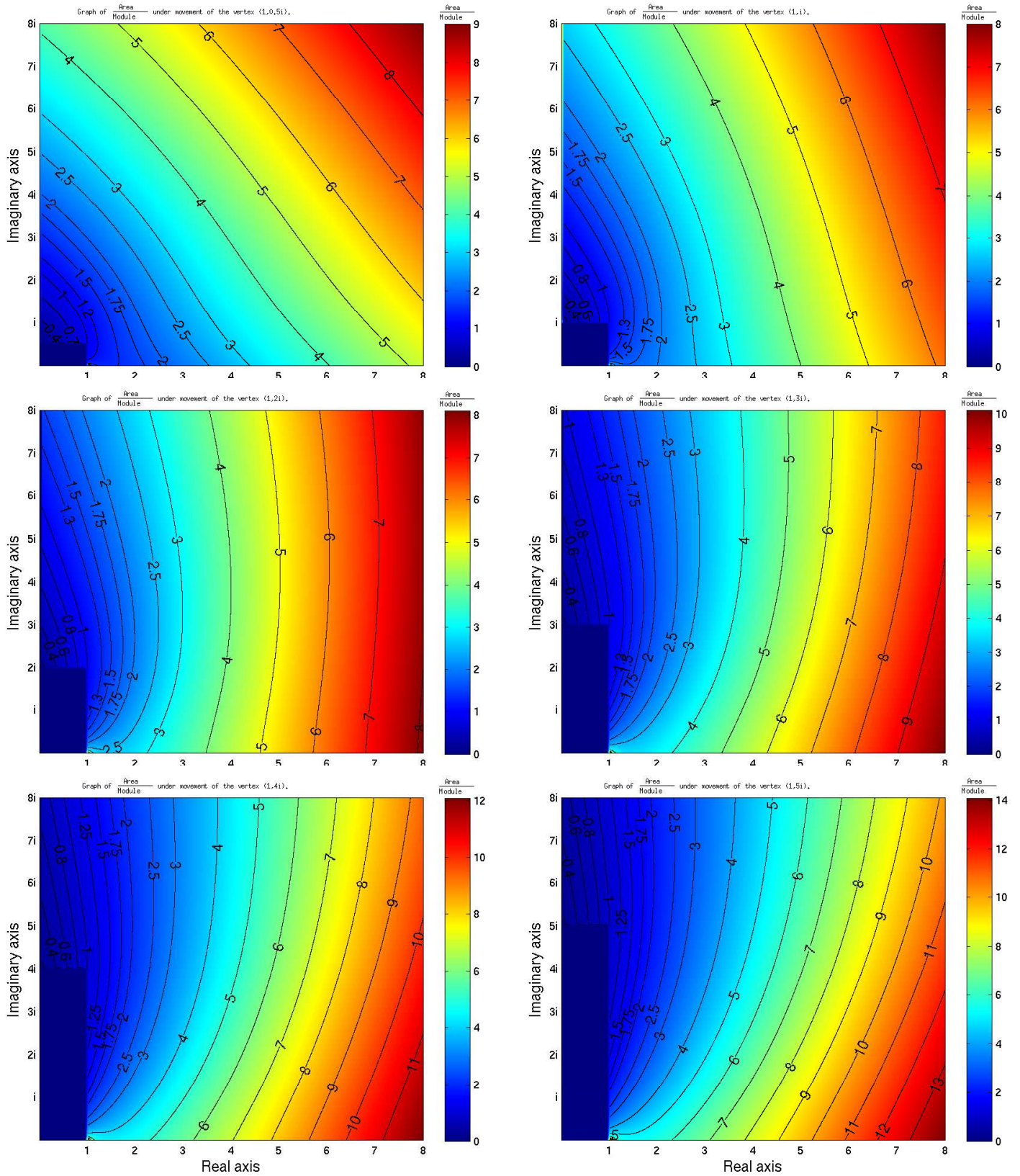


FIGURE 36. Area/Module ratio of a rectangle under movement of one vertex.

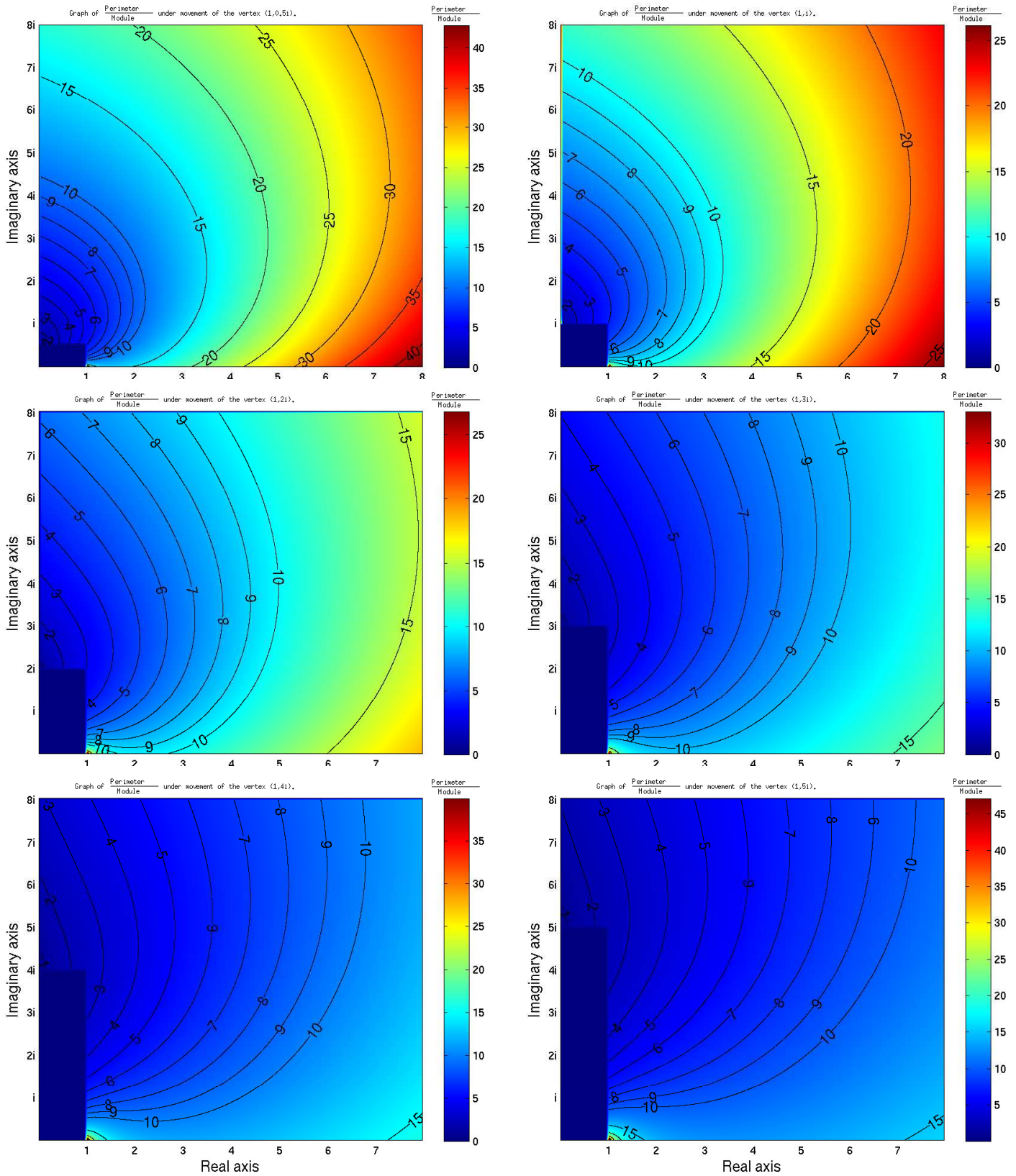


FIGURE 37. Perimeter/Module ratio of a rectangle under movement of one vertex.

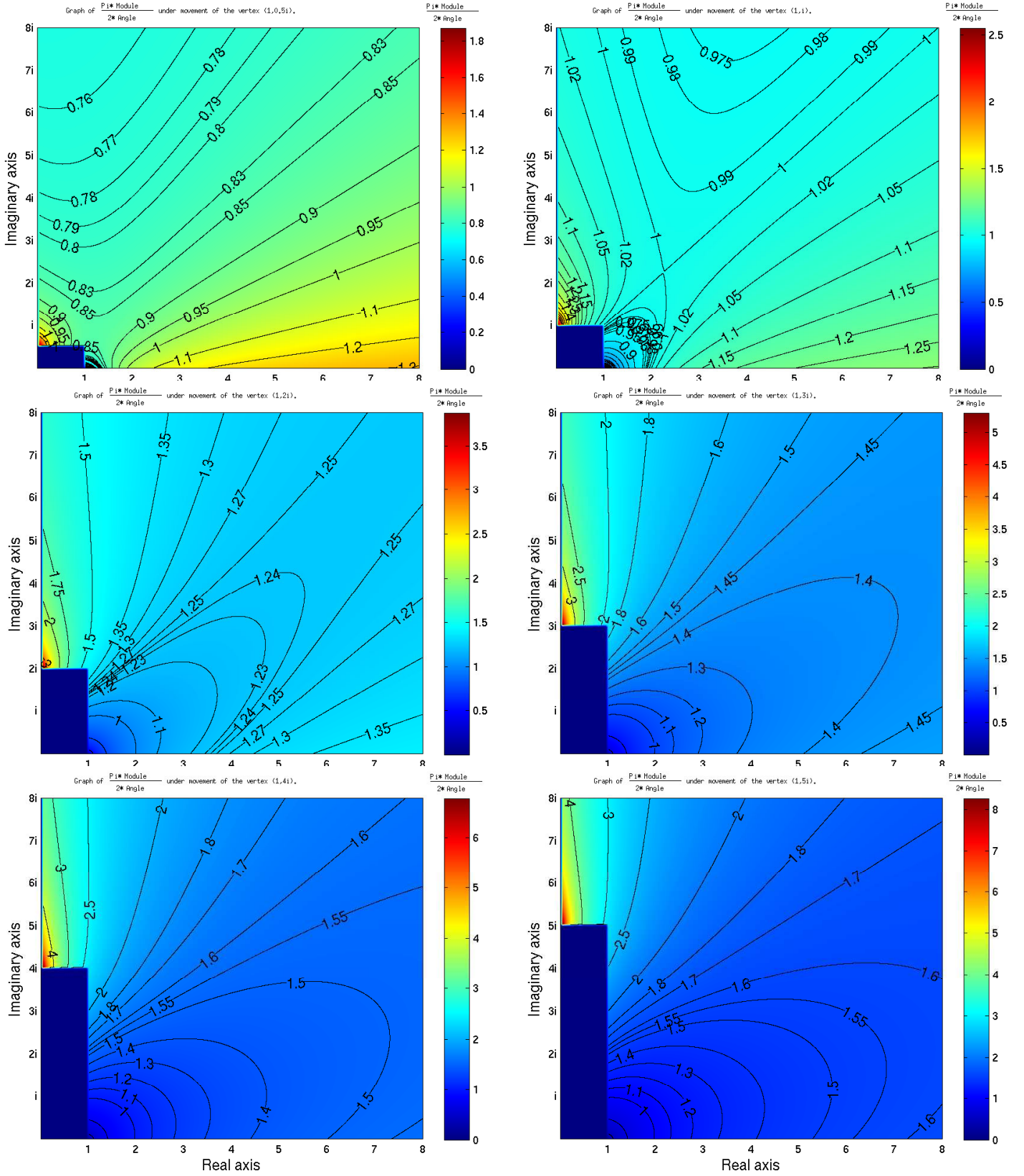


FIGURE 38. $\frac{\pi \cdot \text{Module}}{2 \cdot \alpha}$ ratio of a rectangle under movement of one vertex, where α is the angle between the diagonal lines.

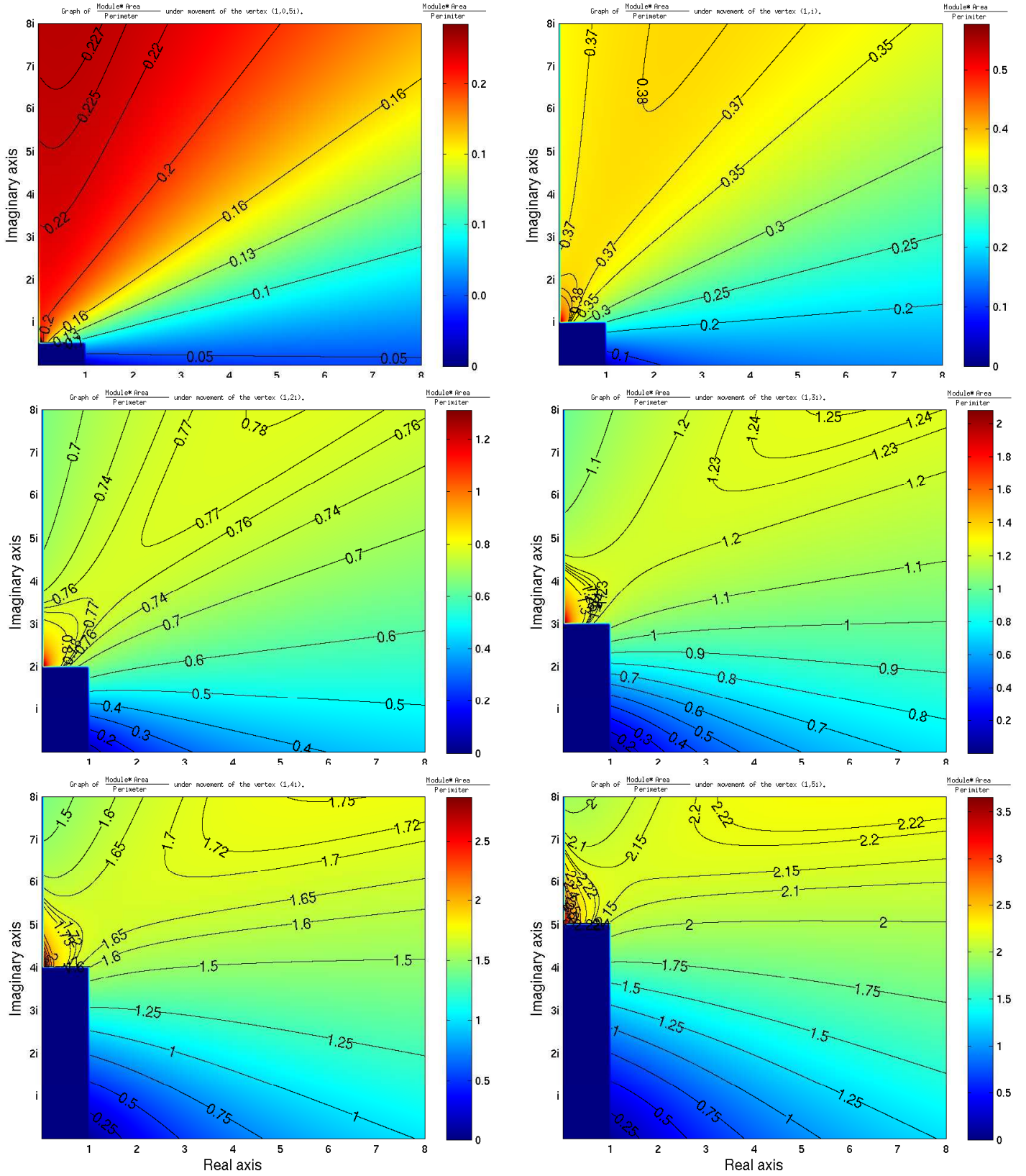


FIGURE 39. $\frac{\text{Module} \cdot \text{Area}}{\text{Perimeter}}$ ratio of a rectangle under movement of one vertex.

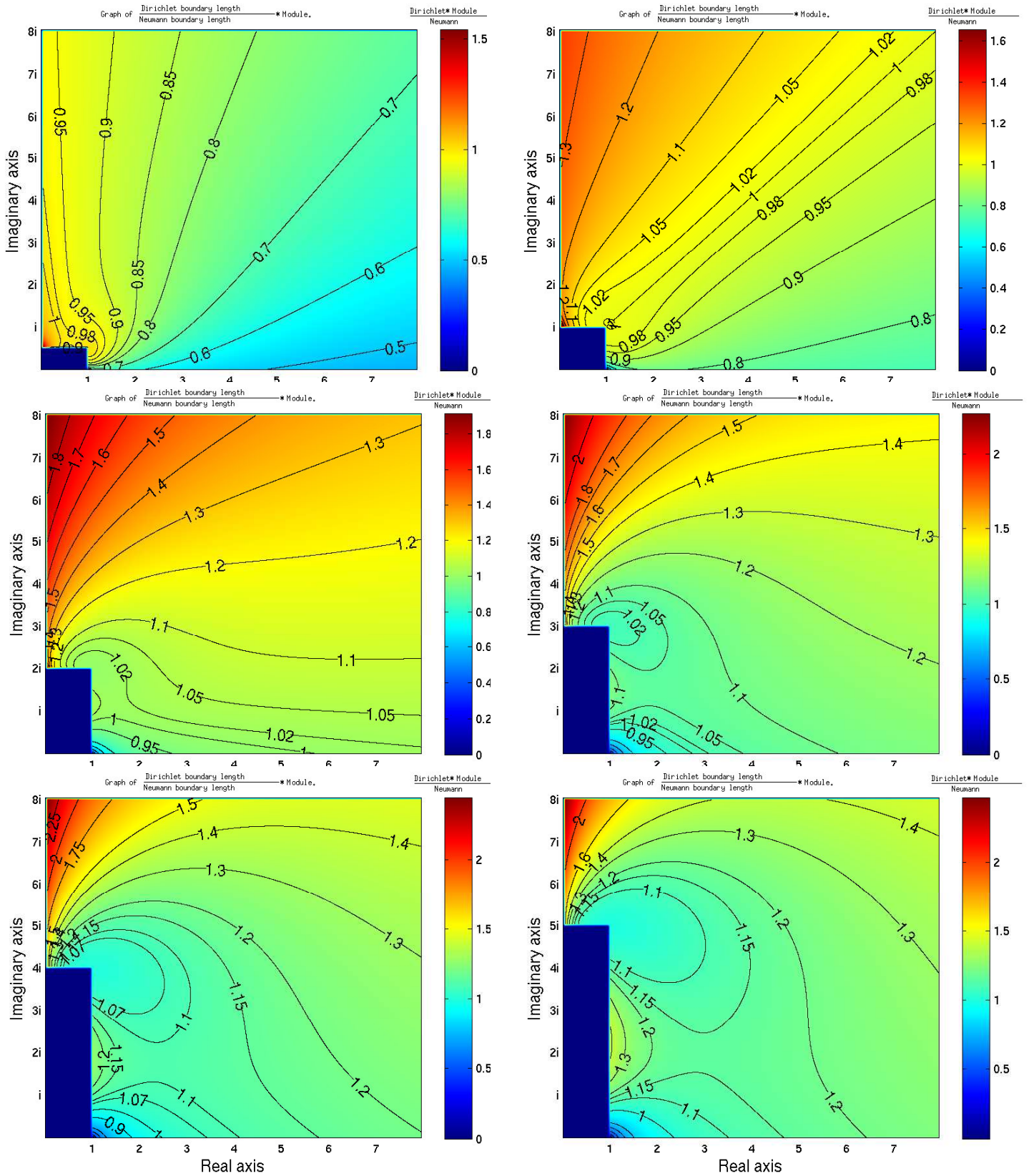


FIGURE 40. $\frac{\text{The Dirichlet boundary length}}{\text{The Neumann boundary length}} * \text{Module}$ ratio of a rectangle under movement of one vertex.

7. OTHER NUMERICAL METHODS

Numerical methods for the computation of the module of a quadrilateral are generally very difficult to implement. Programs often need special functions and other computational tools. This tends to increase program size and programmer's workload, since only small amount of these tools/functions are found ready in language specific libraries. In addition to implementation difficulties, the needed background mathematics are sometimes difficult. What makes this even worse is the fact that some methods are so badly documented, in scientific papers, that is very hard for a novice to even reproduce the same results. Therefore I have come to a conclusion that if someone actually publishes a new numerical algorithm it might be a good idea to publish well documented source code for that algorithm as well. This would increase the probability that someone is actually going to use the algorithm and thus increase knowledge of the algorithm. Another advantage would be that it would allow a writer to omit certain boring details from the scientific paper, while the person who is only interested in using the results would be able to do so. Nowadays this would be very simple to do thanks to the Internet.

In this section we look at some simple methods for the computation of the module of a quadrilateral and then move to more advanced stuff, for more conformal mapping techniques see [Por06]²¹ and [Küh05].

7.1. Challis & Burley iteration.

One of the easiest methods to implement is the Challis & Burley iteration. This iteration was presented by N.V Challis and D.M Burley in 1981[CB82] and it is based on discrete Fourier transformation. This method was then studied by D. Gaier and N. Papamichael and they showed that Challis & Burley iteration was a special case of the method of Garrick[GP87]. They also provided convergence results and made some practical observations. Simplicity of the Challis & Burley iteration comes with a price and the price is that the iteration domains are very restricted and convergence is usually bad.

The basic Challis & Burley iteration works only for quadrilaterals with following domains

$$(7.1) \quad G = \{(x, y) : 0 < x < 1, 0 < y < f(x)\}.$$

where $f(x)$ is arbitrary simple curve[CB82]. We call above domains as Challis & Burley domains²². Note that since we are only interested in computing the module of the quadrilateral we can extend the Challis & Burley domain to include another domain which is formed by reflection.

²¹Antti Rasila has used the Koebe's method, which is introduced in Porter's paper but according to Rasila convergence is slow.

²²Gaier and Papamichael showed that more general Garrick algorithm can be applied to more general domains of the form $G = \{(x, y) : 0 < x < 1, f_1(x) < y < f_2(x)\}$. [GP87]

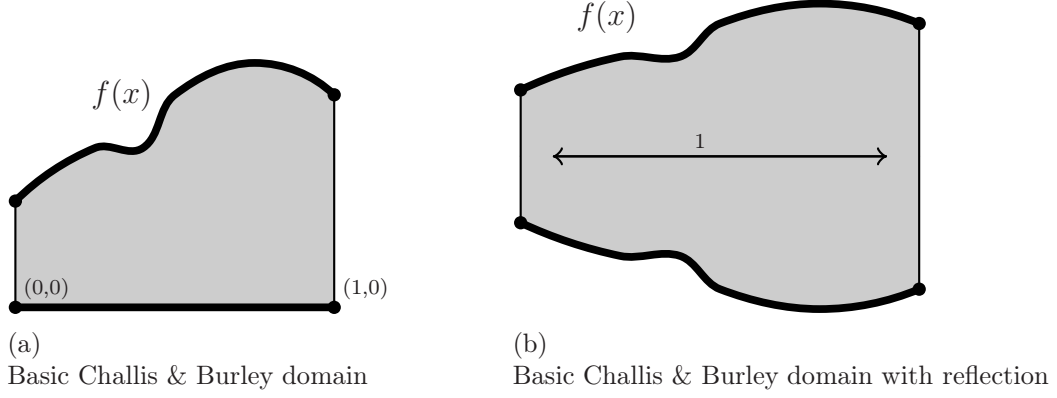


FIGURE 41. Challis & Burley iteration domains. The Dirichlet boundary conditions are marked as thick lines. The Neumann boundary conditions are thin lines.

7.2. Theorem (The basic Challis & Burley iteration). [CB82]

Let G be Challis & Burley domain as above, also divide interval $0 < x < 1$ into N subintervals of equal length. Then the module of the G , $M(G)$ can be calculated as follows

- (1) Estimate values $\gamma_0, \gamma_1, \dots, \gamma_N$. One possibility is to set

$$\gamma_k = \frac{k}{N}$$

- (2) Compute $M(G)$ and b_n by using following formulas

$$\begin{aligned} \frac{1}{M(G)} &= \frac{1}{N} \left(\frac{1}{2} \gamma_0 + \gamma_1 + \dots + \gamma_{N-1} + \frac{1}{2} \gamma_N \right) \\ b_n &= \frac{2}{N} \coth \frac{n\pi}{M(G)} \left(\frac{1}{2} \gamma_0 + \sum_{p=1}^{N-1} \gamma_p \cos \frac{pn\pi}{N} + \frac{1}{2} \gamma_N \cos n\pi \right) \end{aligned}$$

- (3) Compute α_k using following formulas

$$\begin{aligned} \alpha_0 &= 0 & \alpha_N &= 0 \\ \alpha_k &= \sum_{n=1}^N b_n \sin \left(\frac{n\pi k}{N} \right) \end{aligned}$$

- (4) Calculate a new values of $\gamma_0, \gamma_1, \dots, \gamma_N$ by using

$$\gamma_k = f \left(\frac{k}{N} + \alpha_k \right)$$

- (5) Return to (2) and loop for convergence.

Matlab implementation of this algorithm with under-relaxation is in the appendix as well as the algorithm with conjugate series.

According to [CB82][p. 175], Challis & Burley iteration can be improved by utilizing conjugate series. I experimented with this approach (with Simpson's integration rule). It turned out that this is questionable (or there is something wrong with my implementation). More study and testing would be required in order to find out. Now it is unknown if it is better to use conjugate series or not.

Above algorithm converges poorly and typically some under-relaxation must be used in α_k values. It is also difficult to select an optimal under-relaxation value. Gaier and Papamichael suggested a heuristic search algorithm for finding optimal under-relaxation value. Also they noticed that the following quantity

$$(7.3) \quad w = \frac{1}{1 + \left(\sup_{0 \leq x \leq 1} |f'(x)| \right)^2}$$

is usually quite close to the optimal relaxation parameter value, where $f(x)$ is the border curve.[GP87]

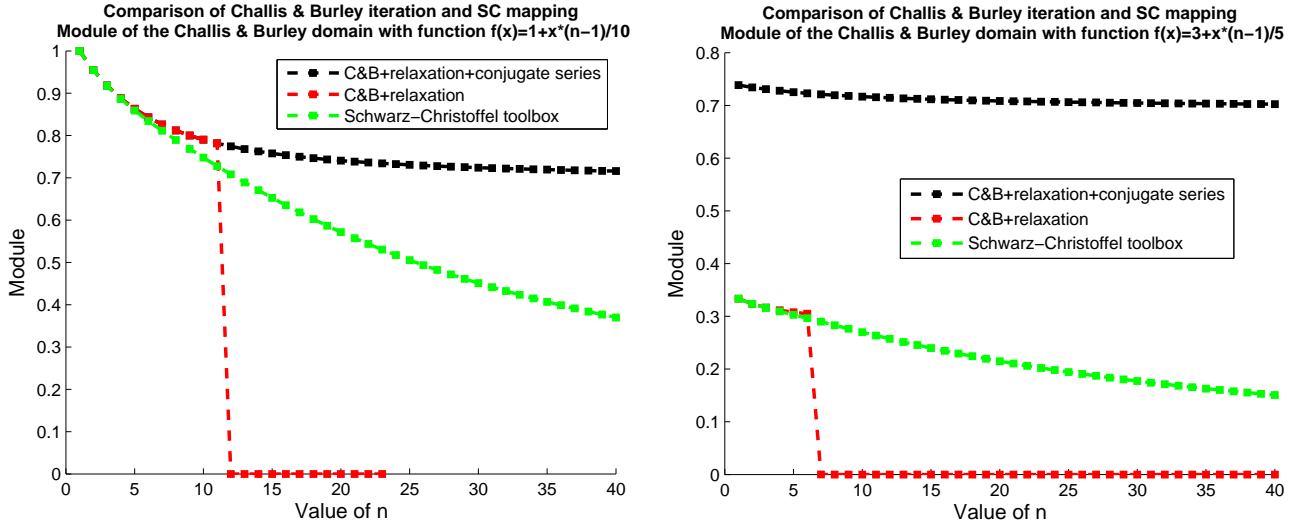


FIGURE 42. Comparison of Challis & Burley iteration and SC mapping. The plot shows the module of the Challis & Burley domain with functions $f(x) = 1 + x * (n - 1)/10$ and $f(x) = 3 + x * (n - 1)/5$. There are 64 division points.

As a rule of thumb, under-relaxation causes even more numerical inaccuracy but it might be the only way to get Challis & Burley iteration to converge. Problems

with Challis & Burley iteration and especially with relaxation are apparent from the Figure 42. The difference in accuracy between two different Challis & Burley versions is due to relaxation value. Limited number of tests suggests that relaxed Challis & Burley behaves similar fashion than conjugate series without relaxation. This suggest that good relaxation values depend on the method and (7.3) is not optimal for conjugate series.

7.2. Heikkala-Vamanamurthy-Vuorinen iteration.

Another method for the module computation is the Heikkala-Vamanamurthy-Vuorinen iteration. This iteration is much more complex than the Challis & Burley iteration. The increase in the complexity comes from the fact that one needs to evaluate hypergeometric functions and elliptic integrals of the first kind. Also some kind of a numerical scheme must be used to find solution of the resulting equation. Originally Newton's iteration was used for this purpose and hence the name iteration.

7.4. Definition (Gaussian hypergeometric function). [HVV04, p. 1]

Given complex numbers a , b and c with $c \neq 0, -1, -2, \dots$, the Gaussian hypergeometric function is the analytic continuation to the slit plane $\mathbb{C}[1, \infty)$ of the series

$$(7.5) \quad F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \quad |z| < 1.$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the shifted factorial function or the Appell symbol

$$(7.6) \quad (a, n) = a(a+1)(a+2) \cdots (a+n-1),$$

for $n \in \mathbb{N}$.

See [LO00] for a comprehensive list of papers on hypergeometric function and computer software used in numerical evaluation.

7.7. Theorem (Heikkala-Vamanamurthy-Vuorinen iteration for the elliptic modulus). [HVV04, p. 6]

Let $0 < a < 1, 0 < b < 1, \max\{a+b, 1\} \leq c \leq 1 + \min\{a, b\}$, and let Q be a quadrilateral in the upper half plane with vertices $0, 1, A$ and B , the interior angles at which are, respectively, $b\pi, (c-b)\pi, (1-a)\pi$ and $(1+a-c)\pi$.²³ Then the elliptic modulus k satisfies the equation

$$(7.8) \quad A = 1 + \frac{Lk'^{2(c-a-b)} F(c-a, c-b; c+1-a-b; k'^2)}{F(a, b; c; k^2)},$$

where $k' = \sqrt{1-k^2}$ and

$$(7.9) \quad L = \frac{B(c-b, 1-a)}{B(b, c-b)} e^{(b+1-c)i\pi}.$$

²³This is a convex quadrilateral. According to Vuorinen there is a chance that this result might hold for non-convex domains too.

For a proof see [HVV04].

In order to use above formula one must choose an initial guess for the k and then iterate for example by Newton's method. When the elliptic modulus k is known, we can calculate the module of a quadrilateral by using equation (4.18), which is repeated below

$$M(Q) = \frac{\mathcal{K}'(k)}{2\mathcal{K}(k)}.$$

Numerical methods for the evaluation of elliptic integrals of the first kind are discussed in the appendix.

7.3. Numerical methods for solving partial differential equations.

There are two classical methods for finding a numerical solution to a partial differential equation, namely the finite difference method(FDM) and the finite element method(FEM). Usually the finite difference method is used for parabolic and hyperbolic partial differential equations and finite element method is used for parabolic and elliptic equations²⁴. Since the Laplace's equation (3.39) is elliptic, finite element methods are generally more suitable for the module of a quadrilateral computations than finite difference methods. One open source finite element solver is Elmer, which has been developed by CSC (Finnish IT centre for science). For additional information and downloads see

<http://www.csc.fi/elmer/>.

The first use of finite element method to compute the capacitance of condensers (and therefore the module) was made by Gerhard Opfer in 1969. The basic finite element method can be enhanced for the capacitance calculations. Betsakos, Samuelsson and Vuorinen suggested applied finite element method(AFEM) in 2004, which according to their publication is reasonably fast and produces very accurate results. The finite element method and AFEM is beyond the scope of this text, for more details see [BSV04] and some finite element introduction text. Rasila and Vuorinen have developed a web-browser based interface for AFEM, unfortunately this web-browser based interface is not available to the public[RVI06].

²⁴A partial differential equation of the form

$$(7.10) \quad a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + d \frac{\partial u}{\partial x} + e \frac{\partial u}{\partial y} + fu = g$$

is called a linear second-order partial differential equation with two variables x and y . If we replace $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ by α and β respectively, we get the following equation

$$(7.11) \quad P(\alpha, \beta) = a\alpha^2 + 2b\alpha\beta + c\beta^2 + d\alpha + e\beta + f$$

The mathematical properties of the solutions of (7.10) are mainly determined by the algebraic properties of the polynomial $P(\alpha, \beta)$. The partial differential equation (7.10) is classified as hyperbolic, parabolic or elliptic according as its discriminant, $b^2 - ac$ is positive, zero or negative.[DZ86, p. 4]

LIST OF SPECIAL SYMBOLS AND ABBREVIATIONS

$\bar{\mathbb{C}}$	The one point compactification of \mathbb{C} , thus $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
$\bar{\mathbb{R}}^2$	The one point compactification of \mathbb{R}^2 , thus $\bar{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$
\mathbb{C}	Set of complex numbers
$\dot{\mathbb{R}}^1$	The two point compactification of \mathbb{R}^1 , thus $\dot{\mathbb{R}}^1 = \mathbb{R}^1 \cup \{\infty\} \cup \{-\infty\}$
Γ	A curve family
γ	A curve
\mathcal{K}'	Complete complementary elliptic integral of the first kind, $\mathcal{K}' = \mathcal{K}(k')$
$\mathcal{K}(k)$	Complete elliptic integral of the first kind
$\mathbf{M}(Q)$	The module of the quadrilateral Q
\mathbb{N}	Set of natural numbers, $\mathbb{N} = \{1, 2, 3, 4, \dots\}$
∂D	∂D is the boundary of D , $\partial D = \bar{D} \setminus D$
\mathbb{R}	Set of real numbers
ρ^*	Extremal metric
\mathbb{Z}	Set of integers, $\mathbb{Z} = \{\dots, -2, -1, -1, 0, 1, 2, \dots\}$
$A_\rho(D)$	ρ -area of domain D
D	A domain is a open connected set in $\bar{\mathbb{R}}^n$ or $\bar{\mathbb{C}}^n$
$d(a, b)$	$d(a, b)$ is a metric, ie the distance from a to b
$F(a, b; c; z)$	Gaussian hypergeometric function
$F(k, \phi)$	Elliptic integral of the first kind
H^+	The upper half-plane, $H^+ = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, y > 0\}$
k	Elliptic modulus
k'	Complementary elliptic modulus, $k' = \sqrt{1 - k^2}$
$L_\rho(\Gamma)$	ρ -length of the curve family γ
$l_\rho(\gamma)$	ρ -length of the curve γ
$m(D, \Gamma)$	Module of the curve family Γ in domain D
<i>odd</i>	Set of odd numbers, <i>odd</i> = $\{1, 3, 5, 7, 9, \dots\}$
$P(\Gamma)$	Set of all admissible metrics for curve family Γ
Q	A quadrilateral
R	A region is an arbitrary set in $\bar{\mathbb{R}}^n$ or $\bar{\mathbb{C}}^n$
S_τ	upper Darboux sum
s_τ	lower Darboux sum
T	A trilateral

APPENDIX A. NUMERICAL EVALUATION OF THE ELLIPTIC INTEGRALS OF THE FIRST KIND

There are several methods for computing elliptic integrals of the first kind. Roughly speaking methods can be divided into two classes: methods for complete integrals only and methods for both complete and incomplete integrals. The oldest methods for the computation of the elliptic integrals are almost as old as the elliptic integral itself. Perhaps the oldest methods are from Gauss in 1799. Gauss discovered following series expansion for the complete elliptic integral of the first kind[MM99, pp. 66-67].

A.1. Theorem (Series expansion for complete elliptic integral of the first kind). [SL99, p. 195]

$$(A.2) \quad \mathcal{K}(k) = \frac{\pi}{2} \left(1 + \left(\frac{1}{2} \right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 + \dots \right)$$

where, k is the elliptic modulus and $0 < k < 1$.

Gauss also discovered that arithmetic and geometric means can be used to evaluate the Gauss's trigonometric version of the complete integral. This method of Gauss was in reality disguised version of Landen's transformation, which was published by English mathematician John Landen in 1775.[MM99, pp. 66-67][EB, V16, p. 154]

A.3. Theorem (Method based on Landen's transformation). [SL99, p. 196]

$$(A.4) \quad F(k, \phi) = \sqrt{\frac{k_1 k_2 k_3 \dots}{k}} \ln \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right),$$

where k is the elliptic modulus and $k < k_1 < k_2 < k_3 < \dots < 1$ and

$$(A.5) \quad k_1 = \frac{2\sqrt{k}}{1+k}, \quad k_2 = \frac{2\sqrt{k_1}}{1+k_1}, \quad \dots, \quad k_{i+1} = \frac{2\sqrt{k_i}}{1+k_i}.$$

The product $k_1 k_2 k_3 \dots$ in (A.4) is considered as infinite product of k_i , in other words $\prod_{i=1}^{\infty} k_i$.

In order to use Landen's transformation to get numerical values, we must fix index i and compute the approximation value for $F(k, \phi)$ with fixed i . Landen's transformation converges reasonable well for elliptic integrals of the first kind.[NRC]

The most modern method currently available is by B. C. Carlson. Carlson introduced new versions of the integrals. These versions are called Carlson's symmetric forms. Carlson's symmetric form for elliptic integral of the first kind is

$$(A.6) \quad R_F(x, y, z) = \frac{1}{2} \int_0^\infty \frac{1}{\sqrt{(t+x)(t+y)(t+z)}} dt$$

and $R_F(x, y, z)$ is related to $\mathcal{K}(k)$ and $F(\phi, k)$ by

$$(A.7) \quad \mathcal{K}(k) = R_F(1, 1 - k^2, 0)$$

$$(A.8) \quad F(\phi, k) = \sin(\phi) R_F\left(\cos^2(\phi), 1 - k^2 \sin^2(\phi), 1\right).$$

There are many versions of Carlson's algorithms but the following algorithms work with complex parameters.

A.9. Theorem (Carlson's algorithm for $\mathcal{K}(k)$). [Car94, p. 11]

Let $x_0 = 1$ and $y_0 = \sqrt{1 - k^2}$. Now iterate by using following formulas

$$(A.10) \quad x_{m+1} = \frac{x_m + y_m}{2} \quad m = 0, 1, 2, 3, \dots,$$

$$(A.11) \quad y_{m+1} = \sqrt{x_m y_m}, \quad m = 0, 1, 2, 3, \dots,$$

and let x_n and y_n be final values of the iteration for fixed m then

$$(A.12) \quad \mathcal{K}(k) \approx \frac{\pi}{x_n + y_n}.$$

A.13. Theorem (Carlson's algorithm for $F(\phi, k)$). [Car94, p. 5]

Set $x_0 = \cos^2(\phi)$, $y_0 = 1 - k^2 \sin^2(\phi)$, $z_0 = 1$ and $A_0 = \frac{x+y+z}{3}$, then iterate for $m = 0, 1, 2, 3, \dots, n$ by using following formulas

$$(A.14) \quad \lambda_m = \sqrt{x_m} \sqrt{y_m} + \sqrt{x_m} \sqrt{z_m} + \sqrt{y_m} \sqrt{z_m}$$

$$(A.15) \quad A_{m+1} = \frac{A_m + \lambda_m}{4}$$

$$(A.16) \quad x_{m+1} = \frac{x_m + \lambda_m}{4}, \quad y_{m+1} = \frac{y_m + \lambda_m}{4}, \quad z_{m+1} = \frac{z_m + \lambda_m}{4}$$

and let A_n be the final value of the iteration for fixed m and then calculate

$$(A.17) \quad X = \frac{A_0 - x_0}{4^n A_n}, \quad Y = \frac{A_0 - y_0}{4^n A_n}, \quad Z = -X - Y,$$

$$(A.18) \quad E_2 = XY - Z^2, E_3 = XYZ$$

Then the value of $F(\phi, k)$ can be calculated from

$$(A.19) \quad F(\phi, k) \approx \sin(\phi) A_n^{-\frac{1}{2}} \left(1 - \frac{1}{10} E_2 + \frac{1}{14} E_3 + \frac{1}{24} E_2^2 - \frac{3}{44} E_2 E_3 \right)$$

See [LO00] for a comprehensive list of papers on the elliptic integrals. The article [LO00] also helps in finding numerical libraries and other software packs for the evaluation of elliptic integrals(or for some other special function).

APPENDIX B. MATLAB PROGRAMS AND SCRIPTS

B.1. Matlab program: Challis & Burley iteration with relaxation.

Challis & Burley iteration for Matlab with Jacobi under relaxation.

```
function mo = cbjor(curveMap, N, accuracy)
%function mo = cb(curveMap, N, accuracy)
%Computes the module of a quadrilateral by using Challis & Burley iteration according to Challis & Burley
%paper + relaxation from the Gaiers and Papamichaels paper. Algorithm is based on discrete fourier transform.
%Note that this iteration works only for the domains of the type (x,y): 0<x<1,0<y<curveMap(x)
%Parameters
%curveMap = mapping that defines the curve that bounds quadrilateral from the above.
%N         = number of division points
%accuracy = Desired accuracy
%Initial values for gammas

%This is the first step of the algorithm
for r=1:(N+1),
    gam(r)=(r-1)/N;
end
err=1+accuracy;

%Calculation of the optical under/over relaxation value
dx=0:0.05:1;
for j=1:length(dx),
    dy(j)=curveMap(dx(j));
end
der=diff(dy)./diff(dx);
tm=max(abs(der));
JOR=(1/(1+tm*tm)); %Relaxation value
counter=1;

%The main iteration loop
while ((counter<250 || err>accuracy) && counter<25000),

    %Calculate a new approximation to the module
    module=N/(sum(gam(2:N))+0.5*gam(1)+0.5*gam(N+1));
    module_old=module;

    %Calculate values of b_n, step 2 continues
    C(1)=1;
    C(2)=cos(pi/N);
    S(1)=0;
    S(2)=sin(pi/N);
    for r=1:N-1,
        C(r+1) = C(2)*C(r) - S(2)*S(r);
        S(r+1) = C(2)*S(r) + S(2)*C(r);
        U(N+1)=0;
        U(N+2)=0;
        for p=N:-1:1,
            U(p)=gam(p+1) + 2*C(r+1)*U(p+1)-U(p+2);
        end
        b(r)=(2/N)*coth((r*pi)/module)*(0.5*gam(1)+C(r+1)*U(1)-U(2)-0.5*gam(N+1)*cos(r*pi));
    end
    C(1)=1;
    C(2)=cos(pi/N);
    S(1)=0;
    S(2)=sin(pi/N);
    counter=counter+1;
    err=err-JOR*err;
end
```

```

%Calculate values of alpha_k, step 3
for r=1:N,
    C(r+1) = C(2)*C(r) - S(2)*S(r);
    S(r+1) = C(2)*S(r) + S(2)*C(r);
    V(N+1)=0;
    V(N+2)=0;
    for q=N-1:-1:1,
        V(q)=b(q) + 2*C(r+1)*V(q+1)-V(q+2);
    end
    alp(r)= V(1)*S(r+1);
end

%Calculate new estimates for gammas, step 4
for k=1:N,
    para=k/N+alp(k);
    gam(k+1)=feval(curveMap,para);
end
gam(1)=feval(curveMap,0);
gam(N+1)=feval(curveMap,1);
counter=counter+1;
err=abs(module-module_old);
end
mo=module;

```

B.2. Matlab program: Challis & Burley iteration with relaxation and cs.
 Challis & Burley iteration for Matlab with Jacobi under relaxation and conjugate series. This program requires Simpson's method, which is given below after the main program.

```
function mo = cbcs(curveMap, N, accuracy)
%function mo = cbcs(curveMap, N, accuracy)
%Computes the module of a quadrilateral by using Challis & Burley iteration with conjugate series according
%to Challis & Burley paper + relaxation from the Gaiers and Papamichaels paper. Algorithm is based on
%discrete fourier transform. Conjugate series uses Simpson's method to integrate.
%Note that this iteration works only for the domains of the type (x,y): 0<x<1,0<y<curveMap(x)
%Parameters
%curveMap = mapping that defines the curve that bounds quadrilateral from the above.
%N         = number of division points
%accuracy = Desired accuracy

%Initial values for gammas
%This is the first step of the algorithm
for r=1:(N+1),
    gam(r)=(r-1)/N;
end
err=100;
counter=1;

%Calculation of the optical under/over relaxation value
dx=0:0.05:1;
for j=1:length(dx),
    dy(j)=curveMap(dx(j));
end
der=diff(dy)./diff(dx);
tm=max(abs(der));
JOR=(1/(1+tm*tm));%Relaxation value

%The main iteration loop
while ((counter<250 || err>accuracy) && counter<25000),

    %Calculate a new approximation to the module
    module=N/(sum(gam(2:N))+0.5*gam(1)+0.5*gam(N+1));
    module_old=module;

    %Calculate values of b_n, step 2 continues
    C(1)=1;
    C(2)=cos(pi/N);
    S(1)=0;
    S(2)=sin(pi/N);
    for r=1:N,
        C(r+1) = C(2)*C(r) - S(2)*S(r);
        S(r+1) = C(2)*S(r) + S(2)*C(r);
        U(N+1)=0;
        U(N+2)=0;
        for p=N:-1:1,
            U(p)=gam(p+1) + 2*C(r+1)*U(p+1)-U(p+2);
        end
        b(r)=(2/N)*coth((r*pi)/module)*(0.5*gam(1)+C(r+1)*U(1)-U(2)-0.5*gam(N+1)*cos(r*pi));
    end
    C(1)=1;
    C(2)=cos(pi/N);
    S(1)=0;
    S(2)=sin(pi/N);

    %Calculate the module
    mo=module;
    err=abs(module - module_old);
    counter=counter+1;
end
```

```

%Calculate values of alpha_k, step 3 (This uses different steps, other
%than mentioned in the text see the original Challis&Burley paper.)
%Using the Simpson's method to calculate the integral
for k=1:N-1,
    gambar=0;
    gambar=0.5*cbsimpson(k,N,gam);
    temp=0;
    for h=1:N-1,
        temp=temp + b(h)*(1-tanh(h*pi/module))*sin(h*pi*k/N);
    end
    alp(k)=-gambar+temp;
end
alp(N)=0;
alp=alp*JOR;

%Calculate new estimates for gammas, step 4
for k=1:N,
    para=k/N+alp(k);
    gam(k+1)=feval(curveMap,para);
end
gam(1)=feval(curveMap,0);
gam(N+1)=feval(curveMap,1);
counter=counter+1;
err=abs(module-module_old);
end
mo=module;

```

Customized Simpson's method, which is needed in the above program.

```

function inte = cbsimpson(k,N,gam)
%function inte =cbsimpson(k,N,gam)
%Simpson's integration method customized for C&B iteration
inte=0;
for r=2:2:N,
    inte=inte+4*cbinteval(k,r-1,N,gam);
end
for r=3:2:N,
    inte=inte+2*cbinteval(k,r-1,N,gam);
end
inte=inte*(1/N)/3;

```

Auxiliary function that is used above.

```

function ival = cbinteval(k,t,N,gam)
if((k+t)<=0)
    ival=gam(abs(k+t)+1);
elseif((k+t)<N+1)
    ival=gam(k+t+1);
else
    ival=gam(2*N+1-t-k);
end
t2=-t;
if((k+t2)<=0)
    ival2=gam(abs(k+t2)+1);
elseif((k+t2)<N+1)
    ival2=gam(k+t2+1);
else
    ival2=gam(2*N+1-t2-k);
end
ival=ival-ival2;
ival=ival*cot(pi*(t/N)*0.5);

```


B.3. Matlab program: Moving vertex data calculation.

The following Matlab function calculates the numerical data for the chapter 6 plots.

```
function mo = computeData(N,L,side)
%N=Number of division points, 151 default
%L=Length of the side, 8 default value
%side=longest side of the rectangle
options = scmapopt('TraceSolution','off') ;
warning('off','MATLAB:conversionToLogical');
xn=1;
yn=1;
Errors=0; %Number of cases when crrectmap fails
iterationsNow=0;%Current iteration counter
Errordata = [0 0 0 0];%Used in case of error to store backup information
Iterationdata =[xn yn iterationsNow];
BeginTime=clock;%Used to calculate the total elapsed time
totaliterations=length(0:(L/N):L)*length(0:(L/N):L);
imagematrix=zeros(N+1,N+1);
sprintf('Completed %d/%d.',iterationsNow,totaliterations)
%main iteration loop and computation
for y=0:(L/N):L,
    for x=0:(L/N):L,
        if (not((x<=1) && (y<=side)))
            try, %Computing the modulus of the quadrilateral
                scMap=crrectmap(polygon([0 1 x+y*i side*i]),[0.5,0.5,0.5,0.5],options);,
                pl2=get(scMap,'rectpolygon');,
                angpl=pl2.angle();,
                angpl=angpl-1;,
                Ipl=find(angpl);,
                valuespl=[pl2.vertex(Ipl(1)) pl2.vertex(Ipl(2)) pl2.vertex(Ipl(3)) pl2.vertex(Ipl(4))];,
                imagematrix(xn,yn)=(max(imag(valuespl)) - min(imag(valuespl)))/(max(real(valuespl)) - min(real(valuespl)));,
                catch,%Computing module of the quadrilateral if crrectmap fails, also storing data in case of crash.
                    Errors=Errors+1,
                    Errordata(Errors+1,1:4)=[x y xn yn];
                    Iterationdata =[xn yn iterationsNow];
                    save iteraatiodata.mat Iterationdata;
                    save kuvamat.mat imagematrix;
                    save errordat.mat Errordata;
                    imagematrix(xn,yn)=modulus(rectmap(polygon([0 1 x+y*i side*i]),[2 3 4 1],options));
                end
            end
            xn=xn+1;
            iterationsNow=iterationsNow+1;
        end
    end
    sprintf('Completed %d/%d.',iterationsNow,totaliterations)
    xn=1;
    yn=yn+1;
    Iterationdata =[xn yn iterationsNow];
    save tempvalues.mat imagematrix; %Saving data in case of crash.
    save iteraatiodata.mat Iterationdata; %Saving the state of the iteration in case of crash.
end
save imagefinal.mat imagematrix;
time=clock-BeginTime;
sprintf('Total elapsed time. Days: %d, Hours: %d, Minutes: %d , Seconds: %d',time(3),time(4),time(5),time(6))
sprintf('Total number of crrectmap failures %d',Errors)
if (Errors>0)
    Errordata
end
mo=imagematrix;
```

B.4. Matlab script: Data plotting.

Unfortunately there is no possibility to publish data plotting scripts for all the figures in chapter 6, however the following Matlab script plots the Figure 39(b), if the right data is in `imagematrix` variable. Very similar scripts have been used to plot the other figures and it is relatively simple to modify the given script for this purpose. See the end of the code listing for further info.

```

N=151;      %Number of division points
L=8;        %Length of the side
side=1;     %Side length of the rectangle

%This script assumes that the original module data is in imagematrix
B=imagematrix;
plotData=zeros(N+1,N+1);
xn=1;
yn=1;
%Iterate the data and calculate area and the perimeter
for y=0:(L/N):L,
    for x=0:(L/N):L,
        if (not((x<=1) && (y<=side)))
            xv = [0; 1; x; 0; 0]; yv = [0; 0; y; side; 0];
            qarea= polyarea(xv,yv); %area of the quadrilateral
            point=[x y];
            corner1=[0 side]; %y-axis point
            corner2=[1 0]; %x-axis point
            diric=1+sqrt(sum((corner1-point).^2)); %The length of the Diriclegt sides
            neumann=side+sqrt(sum( (corner2-point).^2));%The length of the Neumann sides
            plotData(xn,yn)=(qarea/(diric+neumann))*B(xn,yn);
            end
            xn=xn+1;
        end
        xn=1;
        yn=yn+1;
    end

%Use interp2 to get more data points, iscale determines the interpolation level
iscale=4;
A=interp2(plotData,iscale);
%Drawing the figure, pcolor transposes the figure so we transpose it back.
pcolor(transpose(A));
shading interp;
colormap(jet(512));
temp=transpose(A);
[xn yn]=size(temp);
%Drawing the title and the axis of the graph
t=title('Graph of  $\frac{\text{Module}\{\text{Area}\}}{\text{Perimeter}}$  under movement of vertex (1,i).','FontWeight','bold');
set(t,'Interpreter','Latex');
xlabel('Real axis','FontSize',12);
ylabel('Imaginary axis','FontSize',12);
set(gca, 'XTick', 0:xn/8:xn, 'XTickLabel', linspace(0,8,9));
set(gca, 'YTick', 0:yn/8:yn, 'YTickLabel',{'0','i','2i','3i','4i','5i','6i','7i','8i'});
cb=colorbar;
set(get(cb,'Title'),'Interpreter','Latex');
set(get(cb,'Title'),'String','\frac{Module\{Area\}}{Perimeter}');
hold on

```

```

%Modifying the data so that countour lines are drawn properly.
%Removing contour lines from the axis and from the border of the rectangle, since
%they just look bad in the picture. Matlab countour command ingnores NaN points.
for y=1:1:yn,
    for x=1:1:xn,
        if (( y<=(yn/8+4)) && (x<=(side*xn/8+6))),
            temp(x,y)=NaN;
        end
    end
end
%clearing the axis
temp(1:1:xn,1:1:iscale*4)=NaN;
temp(1,1:1:yn)=NaN;

%Drawing countour lines and setting their properties
[Cc,pP]=contour(temp,[0.1 0.2 0.25 0.3 0.35 0.37 0.38 0.39 0.4],'k-');
set(pP,'ShowText','on','LabelSpacing',200,'TextStep',get(pP,'LevelStep')*2)
text_handle = clabel(Cc,pP);
set(text_handle,'FontSize',12,'BackgroundColor','none','Edgecolor','none');

```

Let Q be quadrilateral with one moving vertex be at (x, y) , while other vertices are at $(0, 0)$, $(1, 0)$ and $(0, \text{side})$. Also let us fix the interval $[0, 1]$ to be a Dirichlet boundary, then the following Matlab code calculate the area, side lengths and the angle between the diagonals.

```

%Setting up data vectors for easy use
xv = [0; 1; x; 0; 0]; yv = [0; 0; y; side; 0];
point=[x y]; %The location of the moving vertex
corner1=[0 side]; %The corner of the rectangle located on positive y-axis.
corner2=[1 0]; %The corner of the rectangle located on positive y-axis.

#####
%Calculate the area of the rectangle Q
area = polyarea(xv,yv);

#####
%Calculate side lengths and the perimeter
dirichlet_side_length = 1+sqrt(sum((corner1-point).^2));
neumann_side_length = side+sqrt(sum((corner2-point).^2));

%total perimeter is the sum of all side lengths
perimeter=dirichlet_side_length+neumann_side_length;

#####
%Calculating the angle between the diagonals by using the scalar product.
angle=acos((x-y*side)/(sqrt(x^2+y^2)*sqrt(1+side^2)));

```

APPENDIX C. OTHER FIELDS OF INTEREST

Here is a list of some properties that could be studied further. All these ideas have been thought to some extent, in the process of making this work but they have been rejected because of usability issues or their being too complex and time consuming. Some of these ideas should be considered highly experimental and error prone.

C.1. General concepts and symmetry.

- It would be interesting to categorize domains according to the extremal metric. As we know for a rectangle this extremal metric yields a constant in ρ -area and ρ -length integrals. What kind of domains would we get if we would had x , $x + x^2$ or something else in the integral? Perhaps it would be even possible to derive some summation/other rules for the domains. This might also help to extend the generalized reflection symmetry Theorem to include fractions.
- The module under $\sqrt{\cdot}$ -map. There are some simple domains, which have module $\sqrt{2}$ or $\sqrt{3}$ or something else. Perhaps with this data it would be possible to come up with a model on how the module behaves under $\sqrt{\cdot}$ -map.
- It would be possible to study border rotations by drawing Matlab-figures by using SC toolbox. The required script would be relatively simple. One could try to look for desymmetrization theorem for symmetric quadrilaterals. This theorem would transform symmetric quadrilaterals into nonsymmetric quadrilaterals with symmetric domain. This desymmetrization theorem would explain the Figure 32.
- Derive a inequality(or equality) for the error term in rotation + glue transformation. See Example 5.16.
- Finding general formula for the mass addition to the symmetry axis or to some other place. See Example 5.16.
- Least squares fit into some contour curves, which are given in the numerical results section. One could study especially the generalized symmetry axis.
- Generalization of the reflection principle into domains without straight reflection sides or some other reflection rules for the vertex placement.
- We have only looked at the simplest class of additive quadrilaterals, those that are formed by reflection. However there are probably a lot more such classes. One could start by looking at how non-symmetric boundary conditions with non-symmetrical area increase affect the module.
- It would be interesting to investigate reflections of domains with holes. It might be possible to use reflection property so that reflection to a domain with a hole does not remove the other boundary condition and the domain is still a quadrilateral with a hole and not a ring domain. Perhaps this would be a topic that I would look into if I had more time.

C.2. Numerical methods and Harmonic measure.

- It would be possible to construct a program that calculates the module of a quadrilateral by trying out several different metrics and then selecting the infimum value. This algorithm would produce approximate values in practice.

Although this work did not cover harmonic measure at all, there are some interesting topics that could be studied.

- Stochastically solve the Dirichlet-Neumann problem. There might be two problems in doing this. First, one needs to find the point from which the random walk starts(it is clearly defined though see [Küh05, p. 118]), and secondly, one needs to find the proper module formula for the particular domain in question. In addition to that one must reflect the random walk if it attempts to cross the Neumann sides.
- One topic could be to investigate the connection between the generalized conformal center and the module of a quadrilateral. This topic was presented by Iannaccone in his thesis[Ian03, p. 47] but apparently he did not pursue to study it. Perhaps this point could help picking the right start point for the random walk. The SC toolbox can calculate the conformal center directly so perhaps this topic could be studied even without the harmonic measure.

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